NOTES ON COMPUTABILITY AND ERGODIC THEORY

QIANDU HE

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1. Introduction

This note is devoted to some important definitions and results on computable analysis and ergodic theory. Specifically, for the aspect of computable analysis, we extend the results from the Euclidean space version to the general computable metric space version. For the aspect of ergodic theory, we summarize Rokhlin's formula at the end of this paper.

2. Notation

Let $\mathbb{C}$ be the complex plane and $\widehat{\mathbb{C}}$ be the Riemann sphere. Let $\mathbf{i}$ denote the imaginary unit in the complex plane $\mathbb{C}$. Define the chordal metric σ on $\widehat{\mathbb{C}}$ as follows: $\sigma(z,w) \coloneqq \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$ for all z, $w \in \mathbb{C}$, and $\sigma(\infty,z) = \sigma(z,\infty) \coloneqq \frac{2}{\sqrt{1+|z|^2}}$ for all $z \in \mathbb{C}$. Let S^2 denote an oriented topological 2-sphere. We use $\mathbb{N}$ to denote the set of integers greater than or equal to 1 and $\mathbb{N}^* \coloneqq \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$. We write $\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$ and $\mathbb{N}_0^* \coloneqq \{0\} \cup \mathbb{N}^*$. We denote by $\mathbb{Q}^+$ the set of all positive rational numbers and by $\mathbb{R}^+$ the set of all positive real numbers. The symbol log denotes the natural logarithm. For $x \in \mathbb{R}$, we define $\lfloor x \rfloor$ as the greatest integer $\leqslant x$, $\lceil x \rceil$ the smallest integer $\geqslant x$, and $x^+ \coloneqq \max\{x, 0\}$. The cardinality of a set A is denoted by $\operatorname{card} A$.

Consider a map $f\colon X \to X$ on a set X. We write f^n for the n-th iterate of f, and $f^{-n} \coloneqq (f^n)^{-1}$, for each $n \in \mathbb{N}$. We set $f^0 \coloneqq \mathrm{id}_X$, the identity map on X. For a real-valued function $\phi\colon X \to \mathbb{R}$, we write $S_n\phi(x) = S_n^f\phi(x) \coloneqq \sum_{j=0}^{n-1} \phi\big(f^j(x)\big)$ for $x \in X$ and $n \in \mathbb{N}_0$. We omit the superscript f when the map f is clear from the context. When $n = 0$, by definition $S_0\phi = 0$.

Let (X,d) be a metric space. We denote by $\mathcal{B}(X)$ the σ-algebra of all Borel subsets of X. For each subset $Y \subseteq X$, we denote the diameter of Y by $\operatorname{diam}_d Y \coloneqq \sup\{d(x,y) : x, y \in Y\}$, the interior of Y by $\operatorname{int} Y$, and the characteristic function of Y by $\mathbb{1}_Y$ which maps each $x \in Y$ to $1 \in \mathbb{R}$ and vanishes elsewhere.

For all $r \in \mathbb{R}$ and $x \in X$, we denote the open (resp. closed) ball of radius r centered at x by $B_d(x,r) := \{y \in X : d(x,y) < r\}$ (resp. $\overline{B_d}(x,r) := \{y \in X : d(x,y) \leqslant r\}$). For all $r \in \mathbb{R}$ and non-empty set $K \subseteq X$, we define $d(x,K) := \inf_{y \in K} d(x,y)$, and $B_d(K,r) := \{x \in X : d(x,K) < r\}$. We often omit the metric d in the subscript when it is clear from the context.

For a compact metric space (X,d) and a continuous map $g\colon X \to X$, we denote by $C(X)$ the space of continuous functions from X to $\mathbb{R}$, by $\mathcal{M}(X)$ (resp. $\mathcal{M}(X,g)$) the set of finite signed Borel measures (resp. g-invariant Borel probability measures) on X, and $\mathcal{P}(X)$ the set of Borel probability measures on X. Moreover, for each Borel subset $C \in \mathcal{B}(X)$, $\mathcal{P}(X;C)$ denotes the set $\{\mu \in \mathcal{P}(X) : \mu(C) = 1\}$. By the Riesz representation theorem, we can identify the dual of $C(X)$ with the space $\mathcal{M}(X)$. For $\mu \in \mathcal{M}(X)$, we use $\|\mu\|$ to denote the total variation norm of μ, $\operatorname{supp}\mu$ the support of μ, and

$$\langle \mu, u \rangle := \int u \,\mathrm{d}\mu$$

for each μ-integrable Borel function u on X. If we do not specify otherwise, we equip $C(X)$ with the uniform norm $\|\cdot\|_{C(X)} := \|\cdot\|_\infty$, and equip $\mathcal{M}(X)$, $\mathcal{P}(X)$, and $\mathcal{M}(X,g)$ with the weak* topology.

The space of real-valued Hölder continuous functions with an exponent $\alpha \in (0,1]$ on a compact metric space (X,d) is denoted as $C^{0,\alpha}(X,d)$. For each $\phi \in C^{0,\alpha}(X,d)$,

$$|\phi|_{\alpha,d} := \sup\{|\phi(x) - \phi(y)|/d(x,y)^\alpha : x,\, y \in X,\, x \neq y\}. \tag{2.1}$$

For a complete separable metric space (X,d), we recall the Wasserstein–Kantorovich metric W_d on $\mathcal{P}(X)$ given by

$$W_d(\mu,\nu) := \sup\bigl\{|\langle \mu, f\rangle - \langle \nu, f\rangle| : f \in C^{0,1}(X,d),\, |f|_{1,d} \leqslant 1\bigr\}. \tag{2.2}$$

Note that for Borel probability measures in $\mathcal{P}(X)$, the convergence in W_d is equivalent to the convergence in the weak* topology (see e.g., [Vi09, Corollary 6.13]).

3. Computable analysis

We recall fundamental notions and results from recursion theory and computable analysis.[1] We present, in order, definitions and results concerning the computability of real numbers, computable structures on metric spaces, computability of open sets, functions, compact sets, and probability measures.

3.1. **Computability over the reals.** We begin by reviewing basic notations and concepts from classical recursion theory; for an introduction, see e.g. [Bri94, Chapter 3].

Definition 3.1 (Effective enumeration and recursively enumerable set). Let $S \subseteq \mathbb{N}^*$ be a nonempty set. An *effective enumeration* of S is a sequence $\{x_i\}_{i\in\mathbb{N}}$ with $S = \{x_i : i \in \mathbb{N}\}$ such that there exists an algorithm that, for each $i \in \mathbb{N}$, upon input i, outputs x_i.

Moreover, a set $I \subseteq \mathbb{N}^*$ is said to be a *recursively enumerable set*[2] if $I = \emptyset$ or there exists an effective enumeration of I.

For brevity, the symbol I denotes a nonempty recursively enumerable set throughout this subsection.

Note that $\mathbb{N}^k$, for $k \in \mathbb{N}$, and $\mathbb{N}^*$ are all recursively enumerable sets by Definition 3.1. We then define partial recursive functions and recursive functions.

[1] Our notion of algorithm is consistent with *Type-2 machines* defined in [We00, Definition 2.1.1].

[2] We emphasize that recursively enumerable sets in this article are subsets of $\mathbb{N}^*$.

Definition 3.2 (Partial recursive and recursive function). Let $\{i_n\}_{n\in\mathbb{N}}$ be an effective enumeration of I. We say that $f\colon I \to \mathbb{N}_0^*$ is *partial recursive* if there exists an algorithm that, for each $n \in \mathbb{N}$, on input n, outputs $f(i_n)$ if $f(i_n) \in \mathbb{N}^*$, and runs forever otherwise, namely, if $f(i_n) = 0$. We say that $f\colon I \to \mathbb{N}_0^*$ is *recursive* if f is a partial recursive function with $f(I) \subseteq \mathbb{N}^*$.

We now define the computability of real numbers.

Definition 3.3 (Computable real number). A real number x is called *computable* if there exist three recursive functions $f\colon \mathbb{N} \to \mathbb{N}$, $g\colon \mathbb{N} \to \mathbb{N}$, and $h\colon \mathbb{N} \to \mathbb{N}$ such that $\left|(-1)^{h(n)} f(n)/g(n) - x\right| < 2^{-n}$ for all $i \in I$ and $n \in \mathbb{N}$.

Let $\{x_i\}_{i\in I}$ be a sequence of real numbers. We say that $\{x_i\}_{i\in I}$ is a *sequence of uniformly computable real numbers* if there exist three recursive functions $f\colon \mathbb{N} \times I \to \mathbb{N}$, $g\colon \mathbb{N} \times I \to \mathbb{N}$, and $h\colon \mathbb{N} \times I \to \mathbb{N}$ such that $\left|(-1)^{h(n,i)} f(n,i)/g(n,i) - x_i\right| < 2^{-n}$ for all $i \in I$ and $n \in \mathbb{N}$.

Clearly, $x \in \mathbb{R}$ is computable if and only if $\{x_i\}_{i\in\mathbb{N}}$ defined by $x_i := x$ for all $i \in \mathbb{N}$ is uniformly computable. For analogous concepts in the sequel, we will define the uniform sequence version and regard the individual case as the special case of constant sequences.

3.2. **Computable metric spaces.**

Definition 3.4 (Computable metric space). A *computable metric space* is a triple $(X, \rho, \mathcal{S})$ satisfying that

(i) (X, ρ) is a separable metric space,
(ii) $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$ forms a countable dense subset $\{s_n : n \in \mathbb{N}\}$ of X, and
(iii) $\{\rho(s_m, s_n)\}_{(m,n)\in\mathbb{N}^2}$ is a sequence of uniformly computable real numbers.

The points in $\mathcal{S}$ are called *ideal*. Since $\mathbb{N}^3$ is recursively enumerable, the collection $\mathcal{B} := \{B(s_i, m/n) : i,\, m,\, n \in \mathbb{N}\}$ can be enumerated as $\{B_l\}_{l\in\mathbb{N}}$ satisfying the following: there exists an algorithm that, for each $l \in \mathbb{N}$, upon input l, outputs $i, m, n \in \mathbb{N}$ with $B_l = B(s_i, m/n)$. We call the elements in $\mathcal{B}$ *ideal balls* and such an enumeration of $\mathcal{B}$ an *effective enumeration of ideal balls* in $(X, \rho, \mathcal{S})$.

We then define the computability of points in a computable metric space.

Definition 3.5 (Computable point). Let $(X, \rho, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$, and $\{x_i\}_{i\in I}$ be a sequence of points in X. Then $\{x_i\}_{i\in I}$ is called *uniformly computable* (in $(X, \rho, \mathcal{S})$) if there exists a recursive function $f\colon \mathbb{N} \times I \to \mathbb{N}$ such that $\rho\big(s_{f(n,i)}, x_i\big) < 2^{-n}$ for all $n \in \mathbb{N}$ and $i \in I$. Moreover, a point x in X is *computable* (in $(X, \rho, \mathcal{S})$) if $\{x_i\}_{i\in\mathbb{N}}$ defined by $x_i := x$ for all $i \in \mathbb{N}$ is uniformly computable.

We now specify the computable structure on $\mathbb{R}$. Let $\mathcal{S}_{\mathbb{Q}} = \{q_n\}_{n\in\mathbb{N}}$ be the enumeration of $\mathbb{Q}$ induced by an effective enumeration of $\mathbb{N}^3$ via the mapping $(a, b, c) \mapsto (-1)^c a/b$. Note that $\{d_{\mathbb{R}}(q_m, q_n)\}_{(m,n)\in\mathbb{N}^2}$ is a sequence of uniformly computable real numbers, where $d_{\mathbb{R}}$ is the Euclidean metric. Then the triple $\big(\mathbb{R}, d_{\mathbb{R}}, \mathcal{S}_{\mathbb{Q}}\big)$ forms a computable metric space according to Definition 3.4. A similar construction provides a computable structure for $\mathbb{R}^+$. In this article, we fix these as the standard computability structures on $\mathbb{R}$ and $\mathbb{R}^+$. It is clear that under these structures, Definitions 3.3 and 3.5 are equivalent for the computability of real numbers. That is, a sequence of reals is uniformly computable in one sense if and only if it is in the other.

We also consider a weaker notion of computability over $\mathbb{R}$ that leverages its natural ordered structure.

Definition 3.6 (Semi-computable real number). Let $\{x_i\}_{i\in I}$ be a sequence of real numbers. We say that $\{x_i\}_{i\in I}$ is *uniformly lower* (resp. *upper*) *semi-computable* if there exist three recursive functions $f\colon \mathbb{N} \times I \to \mathbb{N}$, $g\colon \mathbb{N} \times I \to \mathbb{N}$, and $h\colon \mathbb{N} \times I \to \mathbb{N}$ such that for each

$i \in I$, $\left\{(-1)^{h(n,i)} f(n,i)/g(n,i)\right\}_{n\in\mathbb{N}}$ is non-decreasing (resp. non-increasing) and converges to x_i as $n \to +\infty$. Moreover, a real number x is called *lower* (resp. *upper*) *semi-computable* if the sequence $\{x_i\}_{i\in\mathbb{N}}$ defined by $x_i \coloneqq x$ for each $i \in \mathbb{N}$ is uniformly lower (resp. upper) semi-computable.

3.3. **Lower semi-computable open sets.** We define an effective version of open sets and collect some relevant results.

Let $(X, \rho, \mathcal{S})$ be a computable metric space. Let $\mathcal{B}$ be the set of ideal balls, and $\{B_l\}_{l\in\mathbb{N}}$ be an effective enumeration of ideal balls in $(X, \rho, \mathcal{S})$. We define the set $\mathcal{B}_0 \coloneqq \mathcal{B} \cup \{\emptyset\}$ of *extended ideal balls* and an enumeration $\{D_l\}_{l\in\mathbb{N}}$ of $\mathcal{B}_0$ such that $D_1 = \emptyset$ and $D_l = B_{l-1}$ for each integer $l \geqslant 2$. We call such an enumeration an *effective enumeration of extended ideal balls* in $(X, \rho, \mathcal{S})$.

Definition 3.7 (Lower semi-computable open set). Let $(X, \rho, \mathcal{S})$ be a computable metric space, and $\{D_l\}_{l\in\mathbb{N}}$ be an effective enumeration of extended ideal balls. Then a sequence $\{U_i\}_{i\in I}$ of open sets in X is said to be *uniformly lower semi-computable open* (in $(X, \rho, \mathcal{S})$) if there exists a recursive function $f\colon \mathbb{N}\times I \to \mathbb{N}$ such that $U_i = \bigcup_{n\in\mathbb{N}} D_{f(n,i)}$ for each $i \in I$. Moreover, an open set $U \subseteq X$ is called *lower semi-computable open* (in $(X, \rho, \mathcal{S})$) if the sequence $\{U_i\}_{i\in\mathbb{N}}$ defined by $U_i \coloneqq U$ for $i \in \mathbb{N}$ is uniformly lower semi-computable open.

The above definition of a lower semi-computable open set differs slightly from the ones in [BBRY11, Definition 3.4] and [BRY14, Definition 2.4]. In our definition, we use extended ideal balls, which include the empty set. Moreover, the term *recursively open set* in the literature (e.g. [GHR11, Subsection 2.2 and Definition 2.4] and [HR09, Subsection 3.3]) is equivalent to the notion of lower semi-computable open set defined above.

Proposition 3.8. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, and $\{B_n\}_{n\in\mathbb{N}}$ be an effective enumeration of ideal balls in $(X, \rho, \mathcal{S})$. Assume that U_i is an open subset in X for each $i \in I$. Then $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open if and only if there exists a recursively enumerable set $E \subseteq \mathbb{N}\times I$ such that $U_i = \bigcup\{B_n : (n,i) \in E\}$ for each $i \in I$.*

The above result is classical in recursion theory. Here is a brief proof.

Proof of Proposition 3.8. Recall that $D_1 = \emptyset$ and $D_l = B_{l-1}$ for each $l \in \mathbb{N}$. First, we assume that $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets. Then by Definition 3.7, there exists a recursive function $f\colon \mathbb{N}\times I \to \mathbb{N}$ such that $U_i = \bigcup_{n\in\mathbb{N}} D_{f(n,i)}$ for each $i \in I$. Now we define $E \coloneqq \{(f(n,i)-1, i) \in \mathbb{N}\times I : n \in \mathbb{N},\, i \in I,\, f(n,i) \geqslant 2\}$. Thus, $U_i = \bigcup\big\{B_n : n \in \mathbb{N} \text{ and } (n,i) \in E\big\}$ for each $i \in I$. By Definition 3.1, $\mathbb{N}\times I$ is a recursively enumerable set, namely, there exists an effective enumeration $\{(n_m, i_m)\}_{m\in\mathbb{N}}$ of $\mathbb{N}\times I$. Now we define a function $m\colon \mathbb{N} \to \mathbb{N}$ by $m(k) \coloneqq \min\{m \in \mathbb{N} : m > m(k-1),\, f(n_m, i_m) \geqslant 2\}$. Then since f is a recursive function, we obtain that m is a recursive function. Thus, by Definition 3.1, $\{(n_{m(k)}, i_{m(k)})\}_{k\in\mathbb{N}}$ is an effective enumeration of E, hence, E is a recursively enumerable set.

Next, we assume that $E \subseteq \mathbb{N}\times I$ is a recursively enumerable set such that $U_i = \bigcup\big\{B_n : n \in \mathbb{N} \text{ and } (n,i) \in E\big\}$ for each $i \in I$. We split the proof into two cases depending on whether $E = \emptyset$.

Case 1. $E \neq \emptyset$.

In this case, there exists an effective enumeration $\{(n_k, i_k)\}_{k\in\mathbb{N}}$ of the set E, where $n_k \in \mathbb{N}$ and $i_k \in I$ for each $k \in \mathbb{N}$. Then we define a function $f\colon \mathbb{N}\times I \to \mathbb{N}_0$ by $f(n,i) \coloneqq 1 + n_k$, where k is the minimal integer k such that $\operatorname{card}\{m \in \mathbb{N} : 1 \leqslant m \leqslant k \text{ and } i_m = i\} = n$ for all $n \in \mathbb{N}$ and $i \in I$. By Definition 3.2, the function f is recursive. Moreover, by the definition of the function f, $U_i = \bigcup\big\{B_n : n \in \mathbb{N} \text{ and } (n,i) \in E\big\} = \bigcup\big\{D_{n+1} : n \in \mathbb{N} \text{ and } (n,i) \in E\big\} = \bigcup_{n\in\mathbb{N}} D_{f(n,i)}$. Thus, by Definition 3.7, $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets.

Case 2. $E = \emptyset$.

In this case, $U_i = \emptyset$ for each $i \in I$. We define a function $f\colon \mathbb{N} \times I \to \mathbb{N}_0$ by $f(n,i) = dir0o$ for all $n \in \mathbb{N}$ and $i \in I$. By Definition 3.2, f is recursive. Therefore, by Definition 3.7, $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets. □

Note that we can algorithmically decide whether $s \in B$ for each ideal point $s \in \mathcal{S}$ and each extended ideal ball $B \in \mathcal{B}_0$. The following result then follows immediately from Definition 3.7.

Proposition 3.9. *Let $(X, \rho, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$. Assume that $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open. Then there exists a recursively enumerable set $E \subseteq \mathbb{N} \times I$ such that $\{s_n : (n,i) \in E_i\} = \{s_n : n \in \mathbb{N}\} \cap U_i$, where $E_i \coloneqq \{(n,i) \in E : n \in \mathbb{N}\}$ for each $i \in I$.*

Proof. Note that by Definition 3.4 (ii) and (iii), we can decide whether $s \in B$ for each $s \in \mathcal{S}$ and $B \in \mathcal{B}$. Then since $\{B_m\}_{m\in\mathbb{N}}$ is an effective enumeration of the set $\mathcal{B}$ of ideal balls, there exists a recursively enumerable set $F \subseteq \mathbb{N}^2$ such that $\{s_n : (n,m) \in F\} = \{s_n : n \in \mathbb{N}\} \cap B_m$ for each $m \in \mathbb{N}$. Since $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets, by Proposition 3.8, there exists a recursively enumerable set $G \subseteq \mathbb{N} \times I$ such that $U_i = \bigcup\{B_m : (m,i) \in G\}$ for each $i \in I$. Define $E \coloneqq \{(n,i) \in \mathbb{N} \times I : (n,m) \in F,\ (m,i) \in G\}$. Then by the definitions of F and G, we have that for each $i \in I$, $\{s_n : (n,i) \in E\} = \bigcup_{(m,i)\in G}\{s_n : n \in \mathbb{N}\} \cap B_m = \{s_n : n \in \mathbb{N}\} \cap U_i$. □

The following two results are two classical results in computable analysis which both follow immediately from Definitions 3.1 and 3.7.

Proposition 3.10. *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Assume that H and L are two nonempty recursively enumerable sets with $L \subseteq I \times H$, and that $\{U_{i,h}\}_{(i,h)\in L}$ is uniformly lower semi-computable open. Then $\{\bigcup\{U_{i,h} : (i,h) \in L_h\}\}_{h\in H}$ is uniformly lower semi-computable open, where $L_h \coloneqq \{(i,h) \in L : i \in I\}$ for each $h \in H$. In particular, if $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open, then $\bigcup_{i\in I} U_i$ is lower semi-computable open.*

Proof. Since $\{U_{i,k}\}_{(i,k)\in L}$ is a sequence of uniformly lower semi-computable open sets, by Definition 3.8, there exists a recursively enumerable set $E \subseteq \mathbb{N} \times L$ such that $U_{i,k} = \bigcup\{B_n : n \in \mathbb{N},\ (n,i,k) \in E\}$ for each $(i,k) \in L$. Define $F \coloneqq \{(n,k) \in \mathbb{N} \times K : i \in I,\ (n,i,k) \in E\}$. Then F is also a recursively enumerable set. Thus, we obtain that $\bigcup\{U_{i,k} : i \in I,\ (i,k) \in L\} = \bigcup\{B_n : i \in I,\ n \in \mathbb{N},\ (n,i,k) \in E\} = \bigcup\{B_n : n \in \mathbb{N},\ (n,k) \in F\}$ for each $k \in K$. Therefore, $\{\bigcup\{U_{i,k} : i \in I,\ (i,k) \in L\}\}_{k\in K}$ is a sequence of uniformly lower semi-computable open sets.

In particular, we assume that $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets. Let $V_{i,n} \coloneqq U_i$ for all $i \in I$ and $n \in \mathbb{N}$. Then $\{V_{i,n}\}_{(i,n)\in I\times\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. Thus, by previous result, we obtain that $\{\bigcup\{V_{i,n} : i \in I\}\}_{n\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. Note that $\bigcup\{V_{i,n} : i \in I\} = \bigcup_{i\in I} U_i$ for each $n \in \mathbb{N}$. Then by Definition 3.7, $\bigcup_{i\in I} U_i$ is a lower semi-computable open set. □

Proposition 3.11. *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Assume that $\{r_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable real numbers and $\{x_i\}_{i\in I}$ is uniformly computable in $(X, \rho, \mathcal{S})$. Then $\{B(x_i, r_i)\}_{i\in I}$ is uniformly lower semi-computable open.*

Proof. Since $\{r_i\}_{i\in I}$ be a sequence of uniformly lower semi-computable real numbers, by Definition 3.3 (ii) and Definition 3.5, there exists a sequence of uniformly computable real numbers $\{r_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ such that $\{r_{n,i}\}_{n\in\mathbb{N}}$ is non-decreasing and converges to r_i as n converges to $+\infty$ for each $i \in \mathbb{N}$. Since $\{x_i\}_{i\in I}$ is a sequence of uniformly computable points, by Definition 3.5, there exists a recursive function $f\colon \mathbb{N} \times I \to \mathbb{N}$ such that $\rho\big(s_{f(m,i)}, x_i\big) < 2^{-m}$ for all $m \in \mathbb{N}$ and $i \in I$. Thus, we obtain that for each $i \in I$,

$$B(x_i, r_i) = \bigcup_{m\in\mathbb{N}} B\big(s_{f(m,i)}, r_i - 2^{-m}\big) = \bigcup_{(m,n)\in\mathbb{N}^2} B\big(s_{f(m,i)}, r_{n,i} - 2^{-m}\big).$$

Since $f\colon \mathbb{N}\times I\to\mathbb{N}$ is a recursive function, by the uniform computability of the sequence $\{r_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$, we have that $\bigl\{B\bigl(s_{f(m,i)}, r_{n,i}-2^{-m}\bigr)\bigr\}_{(i,m,n)\in I\times\mathbb{N}^2}$ is a sequence of uniformly lower semi-computable open sets. Thus, by Proposition 3.10, $\{B(x_i,r_i)\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets. □

3.4. **Computability of functions.** We begin with the definition of oracles for points.

Definition 3.12 (Oracle). Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space with $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$, and $x\in X$. We say that a function $\tau\colon\mathbb{N}\to\mathbb{N}$ is an *oracle* for x if $\rho\bigl(s_{\tau(n)},x\bigr)<2^{-n}$ for each $n\in\mathbb{N}$.

Proposition 3.13. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. Suppose that $\{i_n\}_{n\in\mathbb{N}}$ is an effective enumeration of a non-empty recursively enumerable set I, and $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets. Then there exists an algorithm that for all $x\in X$, $n\in\mathbb{N}$, and oracle τ for x, on input $n\in\mathbb{N}$ and the oracle $\tau\colon\mathbb{N}\to\mathbb{N}$, halts if and only if $x\in U_{i_n}$.*

Proof. Since $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets, by Definition 3.7, there exists a recursive function $f\colon\mathbb{N}\times I\to\mathbb{N}$ such that $U_i=\bigcup_{k\in\mathbb{N}}D_{f(k,i)}$ for each $i\in I$. Thus, by the definition of $\{D_m\}_{m\in\mathbb{N}}$ and Definition 3.2, there exists an algorithm $\mathcal{A}_0(\cdot,\cdot)$ that on input $k\in\mathbb{N}$ and $i\in I$, outputs $a,\,b\in\mathbb{N}$, and $m\in\mathbb{N}_0$ such that $D_{f(k,i)}=B(s_a,m/b)$.

By Definition 3.12, for each $x\in X$ and each oracle $\tau\colon\mathbb{N}\to\mathbb{N}$, $x\in U_{i_n}$ if and only if $B\bigl(s_{\tau(t)},2^{-t}\bigr)\subseteq D_{f(k,i_n)}$ for some $k\in\mathbb{N}$. With the algorithm $\mathcal{A}_0$, we can compute the centers and radii of these ideal balls. Thus, it is not hard to construct the algorithm that for all $x\in X$, $n\in\mathbb{N}$, and oracle τ for x, on input $n\in\mathbb{N}$ and the oracle $\tau\colon\mathbb{N}\to\mathbb{N}$, halts if and only if $x\in U_{i_n}$. □

With the above definition, computable functions can be defined as follows.

Definition 3.14 (Computable function). Let $(X,\,\rho,\,\mathcal{S})$ and $(X',\,\rho',\,\mathcal{S}')$ be computable metric spaces with $\mathcal{S}=\{s_n\}_{n\in\mathbb{N}}$ and $\mathcal{S}'=\{s'_n\}_{n\in\mathbb{N}}$. Assume that $\{i_n\}_{n\in\mathbb{N}}$ is an effective enumeration of I, and $C_i\subseteq X$ for each $i\in I$. Then a sequence $\{f_i\}_{i\in I}$ of functions $f_i\colon X\to X'$ is called a *sequence of uniformly computable functions with respect to* $\{C_i\}_{i\in I}$ if there exists an algorithm that, for all $l,\,n\in\mathbb{N}$, $x\in C_{i_n}$, and oracle τ for x, on input l, n, and τ, outputs $m\in\mathbb{N}$ with $\rho'(s'_m,f_{i_n}(x))<2^{-l}$. We often omit the phrase "with respect to $\{C_i\}_{i\in I}$" when $C_i=X$ for all $i\in I$. Moreover, a function $f\colon X\to X'$ is said to be a *computable function on* C if $\{f_i\}_{i\in\mathbb{N}}$, defined by $f_i\coloneqq f$ for all $i\in\mathbb{N}$, is a sequence of uniformly computable functions with respect to $\{C_i\}_{i\in\mathbb{N}}$ defined by $C_i\coloneqq C$ for all $i\in\mathbb{N}$. We often omit the phrase "with respect to C" when $C=X$.

Computable functions serve as an effective version of continuous functions. The following result provides examples of computable functions (see e.g. [We00, Examples 4.3.3 and 4.3.13.5]).

Example 3.15. The exponential function $\exp\colon\mathbb{R}\to\mathbb{R}$ and the logarithmic function $\log\colon\mathbb{R}^+\to\mathbb{R}$ are computable functions.

We recall the following classical characterization of computable functions (cf. [RY21a, Proposition 5.2.14] and [BBRY11, Proposition 3.6]).

Before Proposition 3.17, an important notion will be introduced below which is useful in the proof of Propositions 3.17 and 3.19.

Definition 3.16. Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. For each $q\in\mathbb{N}$, we say that a sequence $\{p_i\}_{i=1}^q$ of integers is *admissible* if $\rho\bigl(s_{p_i},s_{p_{i+1}}\bigr)<2^{-i-1}$ for each integer $1\leqslant i\leqslant q-1$. Fix an effective enumeration $\{P_n\}_{n\in\mathbb{N}}$ of all admissible sequences. Moreover, for each admissible sequence $P=\{p_i\}_{i=1}^q$, we can define a corresponding function $\varphi_P\colon\mathbb{N}\to\mathbb{N}$ as follows:

$$\varphi_P(i)\coloneqq\begin{cases}p_i & \text{if } 1\leqslant i\leqslant q;\\ p_q & \text{if } i\geqslant q+1\end{cases}\qquad\text{for each } i\in\mathbb{N}.$$

Indeed, given a computable metric space $(X,\,\rho,\,\mathcal{S})$, by Definition 3.4 (iii), we can check whether a given sequence of finitely many integers is admissible. Hence, by enumerating all sequences of finite integers, it is not difficult to obtain an effective enumeration $\{P_n\}_{n\in\mathbb{N}}$ of all admissible sequences. Moreover, for each admissible sequence P, by Definition 3.12, φ_P is an oracle for $s_{p_q}\in X$, where p_q is the last integer of the admissible sequence P.

Proposition 3.17. *Let $(X,\,\rho,\,\mathcal{S})$ and $(X',\,\rho',\,\mathcal{S}')$ be computable metric spaces. Suppose $\{B'_n\}_{n\in\mathbb{N}}$ is an effective enumeration of ideal balls in $(X',\,\rho',\,\mathcal{S}')$. Given $f_i\colon X\to X'$ and $C_i\subseteq X$ for each $i\in I$, the following statements are equivalent:*

(i) *The sequence $\{f_i\}_{i\in I}$ is a sequence of uniformly computable functions with respect to $\{C_i\}_{i\in I}$.*
(ii) *There exists a sequence $\{U_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ of uniformly lower semi-computable open sets in $(X,\,\rho,\,\mathcal{S})$ such that $f_i^{-1}(B'_n)\cap C_i=U_{n,i}\cap C_i$ for all $i\in I$ and $n\in\mathbb{N}$.*
(iii) *For each nonempty recursively enumerable set K and each sequence $\{V'_k\}_{k\in K}$ of uniformly lower semi-computable open sets, there exists a sequence $\{W_{k,i}\}_{(k,i)\in K\times I}$ of uniformly lower semi-computable open sets in $(X,\,\rho,\,\mathcal{S})$ such that $f_i^{-1}(V'_k)\cap C_i=W_{k,i}\cap C_i$ for all $k\in K$ and $i\in I$.*

Proof. Write $\mathcal{S}=\{s_n\}_{n\in\mathbb{N}}$ and $\mathcal{S}'=\{s'_n\}_{n\in\mathbb{N}}$. Let $\{i_n\}_{n\in\mathbb{N}}$ be an effective enumeration of I. Since $\{B'_n\}_{n\in\mathbb{N}}$ is an effective enumeration of ideal balls, by Definition 3.7, $\{B'_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. Hence, it follows from Proposition 3.10 that statement (ii) is equivalent to statement (iii).

Next, we prove that statement (ii) implies statement (i). Now we design an algorithm $\mathcal{A}_1(\cdot,\,\cdot,\,\cdot)$ that for all $l,\,n\in\mathbb{N}$, $x\in C_{i_n}$, and oracle $\tau\colon\mathbb{N}\to\mathbb{N}$ for x, on input $l,\,n\in\mathbb{N}$, and the oracle τ, outputs $m\in\mathbb{N}$ satisfying that $\rho'(s'_m,f_{i_n}(x))<2^{-l}$, i.e., $x\in f_{i_n}^{-1}\bigl(B_{\rho'}\bigl(s'_m,2^{-l}\bigr)\bigr)$.

Indeed, by Definition 3.4 (ii), $\mathcal{S}'=\{s'_n\}_{n\in\mathbb{N}}$ is dense in X. Thus, for all $l,\,n\in\mathbb{N}$, $x\in C_{i_n}$, there exists corresponding $m\in\mathbb{N}$ with $x\in f_{i_n}^{-1}\bigl(B_{\rho'}\bigl(s'_m,2^{-l}\bigr)\bigr)$. Note that $\bigl\{B_{\rho'}\bigl(s'_m,2^{-l}\bigr)\bigr\}_{(l,m)\in\mathbb{N}^2}$ is a sequence of ideal balls. Hence, it follows from statement (ii) that there exists a sequence $\{V_{l,m,i}\}_{(l,m,i)\in\mathbb{N}^2\times I}$ of uniformly lower semi-computable open sets such that $f_i^{-1}\bigl(B_{\rho'}\bigl(s'_m,2^{-l}\bigr)\bigr)\cap C_i=V_{l,m,i}\cap C_i$ for all $l,\,m\in\mathbb{N}$, and $i\in I$. Hence, for all $i\in I$, $l,\,m\in\mathbb{N}$, and $x\in C_i$, we obtain that $x\in f_i^{-1}\bigl(B_{\rho'}\bigl(s'_m,2^{-l}\bigr)\bigr)$ if and only if $x\in V_{l,m,i}$. Moreover, since $\{V_{l,m,i}\}_{(l,m,i)\in\mathbb{N}^2\times I}$ is uniformly lower semi-computable open, it follows from Proposition 3.13 that we can construct an algorithm that, for all $l,\,m,\,n\in\mathbb{N}$, $x\in C_{i_n}$, and oracle $\tau\colon\mathbb{N}\to\mathbb{N}$ for x, on input $l,\,m,\,n\in\mathbb{N}$ and the oracle τ, halts if and only if $x\in V_{l,m,i_n}$. Thus, we establish that $\{f_i\}_{i\in I}$ is a sequence of uniformly computable functions with respect to $\{C_i\}_{i\in I}$.

Finally, we establish that statement (i) implies statement (ii). Suppose that $\{f_i\}_{i\in I}$ is a sequence of uniformly computable functions with respect to $\{C_i\}_{i\in I}$. We demonstrate that there exists a sequence $\{W_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ of uniformly lower semi-computable open sets such that $f_i^{-1}(B'_n)\cap C_i=W_{n,i}\cap C_i$ for all $n\in\mathbb{N}$ and $i\in I$.

Since $\{f_i\}_{i\in I}$ is a sequence of uniformly computable functions with respect to $\{C_i\}_{i\in I}$, by Definition 3.14, there exists an algorithm $M(\cdot,\cdot,\cdot)$ satisfying that for all $m,\,l\in\mathbb{N}$ and $x\in C_{i_m}$, and oracle $\tau\colon\mathbb{N}\to\mathbb{N}$ for x, $M(m,l,\tau)$ outputs $n\in\mathbb{N}$ satisfying that $\rho'(s'_n,f_{i_m}(x))<2^{-l}$. Let $\{(a_n,b_n)\}_{n\in\mathbb{N}}$ be an effective enumeration of $\mathbb{N}^2$. Now we design an algorithm $M'(\cdot,\cdot)$ that, for all $m,\,n\in\mathbb{N}$, on input $m,\,n\in\mathbb{N}$, outputs a sequence $\{c_t\}_{t\in\mathbb{N}}$ of integers and a sequence $\{r_t\}_{t\in\mathbb{N}}$ of positive rational numbers with $f_{i_m}^{-1}(B'_n)\cap C_{i_m}=\bigl(\bigcup_{t\in\mathbb{N}}B_\rho(s_{c_t},r_t)\bigr)\cap C_{i_m}$.

Begin

(i) Read in the integers m and n.
(ii) Set v and t both to be 1, and $\mathrm{flag}_i=0$ for each $i\in\mathbb{N}$.

(iii) **While** $v \geqslant 1$ **do**

(1) If

(A) $\text{flag}_v = 0$,

(B) the algorithm $M\big(m, b_v, \varphi_{P_{a_v}}\big)$ outputs $n_v \in \mathbb{N}$ satisfying that

$$B_{\rho'}\big(s'_{n_v}, 2^{-b_v}\big) \subseteq B'_n$$

(the algorithm $M\big(m, b_v, \varphi_{P_{a_v}}\big)$ terminates after finitely many steps, and hence the oracle $\varphi_{P_{a_v}}$ is only quired up to some finite precision 2^{-w_v}),

then

(a) the algorithm $M'(m,n)$ outputs $c_t \coloneqq \varphi_{P_{a_v}}(w_v)$ and $r_t \coloneqq 2^{-w_v}$,

(b) set flag_v to be 1, v to be 0, and t to be $t+1$.

(2) Set v to be $v+1$.

End

Finally, by Proposition 3.10, it suffices to show that $f_{i_m}^{-1}(B'_n) \cap C_{i_m} = \big(\bigcup_{t\in\mathbb{N}} B_\rho(s_{c_t}, r_t)\big) \cap C_{i_m}$ for all m, $n \in \mathbb{N}$. Now we fix m, $n \in \mathbb{N}$.

First, we fix $t \in \mathbb{N}$ and show that $B_\rho(s_{c_t}, r_t) \cap C_{i_m} \subseteq f_{i_m}^{-1}(B'_n)$. Indeed, by Step (iii) (1) (a) of the algorithm $M'(m,n)$, we obtain that $c_t = \varphi_{P_{a_v}}(w_v)$ and $r_t = 2^{-w_v}$ for some $v \in \mathbb{N}$ with $B_{\rho'}\big(s'_{n_v}, 2^{-b_v}\big) \subseteq B'_n$. Here n_v is the output of the algorithm $M\big(m, b_v, \varphi_{P_{a_v}}\big)$. Note that $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$ is dense in X. It is not hard to see that, for each $x \in B_\rho(s_{c_t}, r_t) \cap C_{i_m}$, there is a valid oracle $\widetilde{\varphi}_x$ that agrees with $\varphi_{P_{a_v}}$ up to precision 2^{-w_v}. Thus, for each $x \in B_\rho(s_{c_t}, r_t) \cap C_{i_m}$, $M(m, b_v, \widetilde{\varphi}_x)$ outputs the same answer n_v and hence, we must have $f_{i_m}(x) \in B_{\rho'}\big(s'_{n_v}, 2^{-b_v}\big) \subseteq B'_n$. Hence, we have $f_{i_m}\big(B_\rho(s_{c_t}, r_t) \cap C_{i_m}\big) \subseteq B'_n$.

Next, we demonstrate that $\bigcup_{t\in\mathbb{N}} B_\rho(s_{c_t}, r_t) \supseteq f_{i_m}^{-1}(B'_n) \cap C_{i_m}$. Fix a point $x \in f_{i_m}^{-1}(B'_n) \cap C_{i_m}$, and show that $x \in B_\rho(s_{c_t}, r_t)$ for some $t \in \mathbb{N}$. Indeed, since $f_{i_m}(x) \in B'_n$, there exists $l(x) \in \mathbb{N}$ satisfying that $B_{\rho'}\big(f_{i_m}(x), 2^{-l(x)+1}\big) \subseteq B'_n$. Note that $\mathcal{S}$ is dense in X and $x \in C_{i_m}$. It is not hard to see that there is a valid oracle $\overline{\varphi}_x$ for x that satisfies that $\{\overline{\varphi}_x(v)\}_{v=1}^{q}$ is an admissible sequence for each $q \in \mathbb{N}$. Assume that the output of the algorithm $M(m, l(x), \overline{\varphi}_x)$ is $n(x)$. Then by the definition of the algorithm, $\rho'\big(s'_{n(x)}, f_{i_m}(x)\big) < 2^{-l(x)}$. Hence, $B_{\rho'}\big(s'_{n(x)}, 2^{-l(x)}\big) \subseteq B_{\rho'}\big(f_{i_m}(x), 2^{-l(x)+1}\big) \subseteq B'_n$. Assume that the oracle $\overline{\varphi}_x$ is only quired up to the precision $2^{-w(x)}$ by the algorithm $M(m, l(x), \overline{\varphi}_x)$. Denote the sequence $\{\overline{\varphi}_x(v)\}_{v=1}^{w(x)}$ by $Q(x)$. Then $Q(x)$ is an admissible sequence and the oracle $\varphi_{Q(x)}$ agrees with $\overline{\varphi}_x$ up to precision $2^{-w(x)}$. Thus, $M\big(m, l(x), \varphi_{Q(x)}\big)$ outputs the same answer $n(x) \in \mathbb{N}$ as $M(m, l(x), \overline{\varphi}_x)$. Since $Q(x)$ is an admissible sequence, we will run the algorithm $M\big(m, l(x), \varphi_{Q(x)}\big)$ in Step (iii) (1) (B) of the algorithm $M'(m,n)$. Since $B_{\rho'}\big(s'_{n(x)}, 2^{-l(x)}\big) \subseteq B'_n$, in Step (iii) (1) (a) of the algorithm $M'(m,n)$, $M'(m,n)$ will output $c_t = \varphi_{Q(x)}(w(x)) = \overline{\varphi}_x(w(x))$ and $r_t = 2^{-w(x)}$ for some $t \in \mathbb{N}$. Note that $\overline{\varphi}_x$ is an oracle for x. Then we have $x \in B_\rho\big(s_{\overline{\varphi}_x(w(x))}, 2^{-w(x)}\big) = B_\rho(s_{c_t}, r_t)$. □

We now define a notion of weaker computability property for functions.

Definition 3.18 (Semi-computable function). Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, $\{i_n\}_{n\in\mathbb{N}}$ be an effective enumeration of I, and $C_i \subseteq X$ for each $i \in I$. A sequence $\{f_i\}_{i\in I}$ of functions $f_i\colon X \to \mathbb{R}$ is a *sequence of uniformly upper* (resp. *lower*) *semi-computable functions with respect to* $\{C_i\}_{i\in I}$ if there exists an algorithm that, for all l, $n \in \mathbb{N}$, $x \in C_{i_n}$, and oracle τ for x, on input l, n, and τ, outputs $q_{l,n,\tau} \in \mathbb{Q}$ such that for each $n \in \mathbb{N}$, each $x \in C_{i_n}$, and each oracle τ for x, $\{q_{l,n,\tau}\}_{l\in\mathbb{N}}$ is non-increasing (resp. non-decreasing) and converges to $f_{i_n}(x)$ as $l \to +\infty$. We often omit the phrase "with respect to $\{C_i\}_{i\in I}$" when $C_i = X$ for each $i \in I$. Moreover, a function $f\colon X \to \mathbb{R}$ is said to be an *upper* (resp. a *lower*) *semi-computable function* *on* C if $\{f_i\}_{i\in\mathbb{N}}$ defined by $f_i \coloneqq f$ for each $i \in \mathbb{N}$, is a sequence of uniformly upper (resp. lower) semi-computable functions with respect to $\{C_i\}_{i\in\mathbb{N}}$ defined by $C_i \coloneqq C$ for all $i \in \mathbb{N}$. We often omit the phrase "with respect to C" when $C = X$.

The following proposition is an immediate consequence of Proposition 3.17.

Proposition 3.19. *Let* $(X, \rho, \mathcal{S})$ *be a computable metric space, and* $\mathcal{S}_{\mathbb{Q}} = \{q_n\}_{n\in\mathbb{N}}$. *Given* $f_i \colon X \to \mathbb{R}$ *and* $C_i \subseteq X$ *for all* $i \in I$*, the following statements are equivalent:*

(i) *The sequence* $\{f_i\}_{i\in I}$ *is a sequence of uniformly upper (resp. lower) semi-computable functions with respect to* $\{C_i\}_{i\in I}$.

(ii) *There exists a sequence* $\{U_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ *of uniformly lower semi-computable open sets in* $(X, \rho, \mathcal{S})$ *such that* $f_i^{-1}(Q_n)\cap C_i = U_{n,i}\cap C_i$ *with* $Q_n \coloneqq (-\infty, q_n)$ *(resp.* $Q_n \coloneqq (q_n, +\infty)$*) for all* $i \in I$ *and* $n \in \mathbb{N}$.

(iii) *For each nonempty recursively enumerable set* L *and each sequence* $\{r_l\}_{l\in L}$ *of uniformly computable real numbers, there exists a sequence* $\{W_{l,i}\}_{(l,i)\in L\times I}$ *of uniformly lower semi-computable open sets in* $(X, \rho, \mathcal{S})$ *such that* $f_i^{-1}(R_l)\cap C_i = W_{l,i}\cap C_i$ *with* $R_l \coloneqq (-\infty, r_l)$ *(resp.* $R_l \coloneqq (r_l, +\infty)$*) for all* $l \in L$ *and* $i \in I$.

Proof. Let $\{i_n\}_{n\in\mathbb{N}}$ be an effective enumeration of I. It follows from Proposition 3.10 that statement (ii) is equivalent to statement (iii). Hence, it suffices to show that statement (i) is equivalent to statement (ii). Moreover, by Definition 3.18, $\{f_i\}_{i\in I}$ is a sequence of uniformly upper semi-computable functions with respect to $\{C_i\}_{i\in I}$ if and only if $\{-f_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable functions with respect to $\{C_i\}_{i\in I}$. Hence, it suffices to verify the case where $\{f_i\}_{i\in I}$ is a sequence of uniformly upper semi-computable functions with respect to $\{C_i\}_{i\in I}$.

(ii) $\Rightarrow$ (i): Assume that there exists a sequence $\{U_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ of uniformly lower semi-computable open sets in $(X, \rho, \mathcal{S})$ such that $f_i^{-1}(Q_n)\cap C_i = U_{n,i}\cap C_i$ with $Q_n \coloneqq (-\infty, q_n)$ for all $i \in I$ and $n \in \mathbb{N}$. Then we construct an algorithm as follows. First, the algorithm will read in the input l, $n \in \mathbb{N}$ and the oracle $\tau \colon \mathbb{N} \to \mathbb{N}$ for the point $x \in C_{i_n}$. Applying Proposition 3.13, we can find l different integers $m_1, m_2, \dots, m_l$ with $x \in U_{m_k,i_n}$ for each integer $1 \leqslant k \leqslant l$. In this case, the algorithm will output $\min\{q_{m_1}, q_{m_2}, \dots, q_{m_l}\} \in \mathbb{Q}$.

Now we verify that such algorithm ensures that $\{f_i\}_{i\in I}$ is a sequence of uniformly upper semi-computable functions with respect to $\{C_i\}_{i\in I}$. Indeed, for all $i \in I$ and $n \in \mathbb{N}$, since $f_i^{-1}(Q_n)\cap C_i = U_{n,i}\cap C_i$ and $Q_n = (-\infty, q_n)$, we obtain that for each $x \in C_i$, $x \in U_{n,i}$ is equivalent to $f_i(x) < q_n$. Combined with the definition of the above algorithm, this implies that for fixed $n \in \mathbb{N}$ and oracle $\tau \colon \mathbb{N} \to \mathbb{N}$ for $x \in C_{i_n}$, the sequence of outputs of the algorithm on input $l \in \mathbb{N}$ is non-increasing and converges to $f_{i_n(x)}$. Thus, by Definition 3.18, $\{f_i\}_{i\in I}$ is a sequence of uniformly upper semi-computable functions with respect to $\{C_i\}_{i\in I}$.

(i) $\Rightarrow$ (ii): Assume that $\{f_i\}_{i\in I}$ is a sequence of uniformly upper semi-computable functions with respect to $\{C_i\}_{i\in I}$. Then by Definition 3.18, there exists an algorithm $M(\cdot,\cdot,\cdot)$ that, for all l, $n \in \mathbb{N}$, $x \in C_{i_n}$, and oracle τ for x, on input l, n, and τ, outputs $q_{l,n,\tau} \in \mathbb{Q}$ such that for each $n \in \mathbb{N}$, each $x \in C_{i_n}$, and each oracle τ for x, $\{q_{l,n,\tau}\}_{l\in\mathbb{N}}$ is non-increasing (resp. non-decreasing) and converges to $f_{i_n}(x)$ as $l \to +\infty$. Let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be an effective enumeration of $\mathbb{N}^2$. Now we design an algorithm $M'(\cdot,\cdot)$ that, for all m, $n \in \mathbb{N}$, on input m, $n \in \mathbb{N}$, outputs a sequence $\{c_t\}_{t\in\mathbb{N}}$ of integers and a sequence $\{r_t\}_{t\in\mathbb{N}}$ of positive rational numbers with $f_{i_n}^{-1}(Q_m)\cap C_{i_n} = \left(\bigcup_{t\in\mathbb{N}} B_\rho(s_{c_t}, r_t)\right)\cap C_{i_n}$.

Begin

(i) Read in the integers m and n.

(ii) Set v and t both to be 1, and $\text{flag}_i = 0$ for each $i \in \mathbb{N}$.

(iii) **While** $v \geqslant 1$ **do**

(1) If

(A) $\text{flag}_v = 0$,

(B) the algorithm $M\big(b_v, n, \varphi_{P_{a_v}}\big)$ outputs $q \in \mathbb{Q}$ with $q \leqslant q_m$
(the algorithm $M\big(b_v, n, \varphi_{P_{a_v}}\big)$ terminates after finitely many steps, and hence the oracle $\varphi_{P_{a_v}}$ is only quired up to some finite precision 2^{-w_v}),

then

(a) the algorithm $M'(m, n)$ outputs $c_t \coloneqq \varphi_{P_{a_v}}(w_v)$ and $r_t \coloneqq 2^{-w_v}$,

(b) set flag_v to be 1, v to be 0, and t to be $t + 1$.

(2) Set v to be $v + 1$.

End

Finally, by Proposition 3.10, it suffices to show that $f_{i_n}^{-1}(Q_m) \cap C_{i_n} = \big(\bigcup_{t\in\mathbb{N}} B_\rho(s_{c_t}, r_t)\big) \cap C_{i_n}$ for all m, $n \in \mathbb{N}$. Now we fix m, $n \in \mathbb{N}$.

First, we fix $t \in \mathbb{N}$ and show that $B_\rho(s_{c_t}, r_t) \cap C_{i_n} \subseteq f_{i_n}^{-1}(Q_m)$. Indeed, by Step (iii) (1) (a) of the algorithm $M'(m, n)$, we obtain that $c_t = \varphi_{P_{a_v}}(w_v)$ and $r_t = 2^{-w_v}$ for some $v \in \mathbb{N}$ with $q \leqslant q_m$, where q is the output of the algorithm $M\big(b_v, n, \varphi_{P_{a_v}}\big)$. Note that $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$ is dense in X. It is not hard to see that, for each $x \in B_\rho(s_{c_t}, r_t) \cap C_{i_n}$, there is a valid oracle $\widetilde{\varphi}_x$ that agrees with $\varphi_{P_{a_v}}$ up to precision 2^{-w_v}. Thus, for each $x \in B_\rho(s_{c_t}, r_t) \cap C_{i_n}$, $M(b_v, n, \widetilde{\varphi}_x)$ outputs the same answer $q \in \mathbb{Q}$ and hence, we must have $f_{i_n}(x) \in (-\infty, q) \subseteq (-\infty, q_m)$. Hence, we have $f_{i_n}\big(B_\rho(s_{c_t}, r_t) \cap C_{i_n}\big) \subseteq (-\infty, q_m)$.

Next, we demonstrate that $\bigcup_{t\in\mathbb{N}} B_\rho(s_{c_t}, r_t) \supseteq f_{i_n}^{-1}((-\infty, q_m)) \cap C_{i_n}$. Fix $x \in f_{i_n}^{-1}((-\infty, q_m)) \cap C_{i_n}$, and show that $x \in B_\rho(s_{c_t}, r_t)$ for some $t \in \mathbb{N}$. Indeed, since $\mathcal{S}$ is dense in X and $x \in C_{i_m}$. It is not hard to see that there is a valid oracle $\overline{\varphi}_x$ for x that satisfies that $\{\overline{\varphi}_x(v)\}_{v=1}^{q}$ is an admissible sequence for each $q \in \mathbb{N}$. Moreover, since $f_{i_n}(x) \in (-\infty, q_m)$, by the definition of the algorithm $\mathcal{M}(\cdot, \cdot, \cdot)$, there exists $l_0 \in \mathbb{N}$ satisfying that $M(l_0, n, \overline{\varphi}_x)$ outputs $r_0 \in \mathbb{Q}$ with $r_0 \leqslant q_m$. Thus, by the definition of the algorithm, we have $f_{i_n}(x) \leqslant r_0$. Assume that the oracle $\overline{\varphi}_x$ is only quired up to the precision 2^{-w_0} by the algorithm $M(l_0, n, \overline{\varphi}_x)$. Denote the sequence $\{\overline{\varphi}_x(v)\}_{v=1}^{w_0}$ by V. Then V is an admissible sequence and the oracle φ_V agrees with $\overline{\varphi}_x$ up to precision 2^{-w_0}. Thus, $M\big(l_0, n, \varphi_V\big)$ outputs the same answer $r_0 \in \mathbb{Q}$ as $M(l_0, n, \overline{\varphi}_x)$. Since V is an admissible sequence, we will run the algorithm $M\big(l_0, n, \varphi_V\big)$ in Step (iii) (1) (B) of the algorithm $M'(m, n)$. Since $r_0 \leqslant q_m$, in Step (iii) (1) (a) of the algorithm $M'(m, n)$, $M'(m, n)$ will output $c_t = \varphi_V(w_0) = \overline{\varphi}_x(w_0)$ and $r_t = 2^{-w_0}$ for some $t \in \mathbb{N}$. Note that $\overline{\varphi}_x$ is an oracle for x. Then we have $x \in B_\rho\big(s_{\overline{\varphi}_x(w_0)}, 2^{-w_0}\big) = B_\rho(s_{c_t}, r_t)$. □

As an immediate corollary, we obtain the following result.

Corollary 3.20. *A sequence of real-valued functions is a sequence of uniformly computable functions if and only it is simultaneously a sequence of uniformly lower semi-computable functions and a sequence of upper semi-computable functions. Consequently, a sequence of real numbers is a sequence of uniformly computable real numbers if and only if it is simultaneously a sequence of uniformly lower semi-computable real numbers and a sequence of uniformly upper semi-computable real numbers.*

In this corollary, the statement on functions follows from Propositions 3.17 and 3.19. Considering constant functions, we have the statement on real numbers (cf. [BBRY11, Proposition 3.3]).

3.5. **Recursively compact sets and recursively precompact metric spaces.** Here we recall the definitions of recursive compactness and recursive precompactness. For a more detailed discussion, see [GHR11, Section 2].

Definition 3.21 (Recursively compact set). Let $(X, \rho, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$, and $\{i_l\}_{l\in\mathbb{N}}$ be an effective enumeration of I. A sequence $\{K_i\}_{i\in I}$ of compact sets in X is called *uniformly recursively compact* (in $(X, \rho, \mathcal{S})$) if there exists an algorithm that, for each $n \in \mathbb{N}$, each sequence $\{m_n\}_{n=1}^{p}$ of integers, and each sequence $\{q_n\}_{n=1}^{p}$ of positive rational numbers, upon input, halts if and only if $K_{i_l} \subseteq \bigcup_{n=1}^{p} B(s_{m_n}, q_n)$. Moreover, a set $K \subseteq X$ is called *recursively compact* (in $(X, \rho, \mathcal{S})$) if the sequence $\{K_i\}_{i\in\mathbb{N}}$ defined by $K_i := K$ for each $i \in \mathbb{N}$, is uniformly recursively compact.

Note that for each compact set K and each function $f\colon \mathbb{N} \to \mathbb{N}$, $K \subseteq \bigcup_{n\in\mathbb{N}} D_{f(n)}$ if and only if $K \subseteq \bigcup_{n=1}^{k} D_{f(n)}$ for some $k \in \mathbb{N}$. This implies the following result.

Proposition 3.22. *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Suppose $\{h_m\}_{m\in\mathbb{N}}$ (resp. $\{l_n\}_{n\in\mathbb{N}}$) is an effective enumeration of a nonempty recursively enumerable set H (resp. L). Assume that $\{K_h\}_{h\in H}$ is uniformly recursively compact and $\{U_l\}_{l\in L}$ is uniformly lower semi-computable open. Then there exists an algorithm that, for all m, $n \in \mathbb{N}$, upon input, halts if and only if $K_{h_m} \subseteq U_{l_n}$.*

We collect some fundamental properties of recursively compact sets (cf. [GHR11, Propositions 1 & 3]).

Proposition 3.23. *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Assume that X is recursively compact, and $\{K_i\}_{i\in I}$ is uniformly recursively compact. Then the following statements are true:*

(i) *Let $x_i \in X$ for each $i \in I$. Then $\{x_i\}_{i\in I}$ is uniformly computable if and only if the sequence $\{\{x_i\}\}_{i\in I}$ of singletons is uniformly recursively compact.*
(ii) *$\{X \smallsetminus K_i\}_{i\in I}$ is uniformly lower semi-computable open.*
(iii) *If $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open, then $\{K_i \smallsetminus U_i\}_{i\in I}$ is uniformly recursively compact.*
(iv) *If $\{f_i\}_{i\in I}$ is a sequence of uniformly lower (resp. upper) semi-computable functions $f_i\colon X \to \mathbb{R}$ with respect to $\{K_i\}_{i\in I}$, then $\{\inf_{x\in K_i} f_i(x)\}_{i\in I}$ (resp. $\{\sup_{x\in K_i} f_i(x)\}_{i\in I}$) is uniformly lower (resp. upper) semi-computable.*
(v) *If $\{T_i\}_{i\in I}$ is a sequence of uniformly computable functions $T_i\colon X \to X$ with respect to $\{K_i\}_{i\in I}$, then $\{T_i(K_i)\}_{i\in I}$ is uniformly recursively compact.*

Proof. (i) Let $\{i_n\}_{n\in\mathbb{N}}$ be an effective enumeration of I. First, we prove the forward implication. Assume that $\{x_i\}_{i\in I}$ is uniformly computable, then there exists a recursive function $\varphi\colon \mathbb{N}\times I \to \mathbb{N}$ such that $\rho\bigl(s_{\varphi(n,i)}, x_i\bigr) < 2^{-n}$ for all $n \in \mathbb{N}$ and $i \in I$. Hence, it follows from Proposition 3.13 and Definition 3.21 that $\{\{x_i\}\}_{i\in I}$ of singletons is uniformly recursively compact. Conversely, we assume that $\{\{x_i\}\}_{i\in I}$ is uniformly recursively compact. Hence, by Definition 3.21, there exists an algorithm $\mathcal{A}_0(l, m, n)$ that, on input l, m, $n \in \mathbb{N}$, halts if and only if $x_{i_n} \in B(s_l, 2^{-m})$. Let $\mathcal{A}(t, m, n)$ be the algorithm that, on input t, m, $n \in \mathbb{N}$, outputs the minimal integer $1 \leqslant l \leqslant t$ such that $\mathcal{A}_0(l, m, n)$ halts before its $(t + 1 - l)$-th steps if such l exists, and outputs 0 otherwise. We run $\mathcal{A}(t, m, n)$ for all t, m, $n \in \mathbb{N}$ one by one until we find $t(m, n) \in \mathbb{N}$ with $\mathcal{A}(t(m, n), m, n)$ does not output 0. Note that $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$ is dense in X. Then $t(m, n)$ exists for all m, $n \in \mathbb{N}$. Let $f(m, n)$ be the output of $\mathcal{A}(t(m, n), m, n)$. It is not hard to see that the function $f\colon \mathbb{N}^2 \to \mathbb{N}$ defined above is recursive. Moreover, by the definition of f, we obtain that $x_{i_n} \in B\bigl(s_{f(m,n)}, 2^{-m}\bigr)$ for all m, $n \in \mathbb{N}$. Hence, by Definition 3.5, $\{x_i\}_{i\in I}$ is uniformly computable.

(ii) Recall that $\{B_l\}_{l\in\mathbb{N}}$ is the effective enumeration of ideal balls in $(X, \rho, \mathcal{S})$ and write $B_l = B\bigl(s_{n_l}, r_l\bigr)$ for each $l \in \mathbb{N}$. First, we define $C_l := \bigl\{x \in X : \rho\bigl(x, s_{n_l}\bigr) > r_l\bigr\}$ and $f_l(x) = \rho\bigl(x, s_{n_l}\bigr)$ for all $x \in X$ and $l \in \mathbb{N}$. Then $\{f_l\}_{l\in\mathbb{N}}$ is a sequence of uniformly computable functions. By Definition 3.7, $\{(r_l, +\infty)\}_{l\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. Combined with the fact that $C_l = f_l^{-1}((r_l, +\infty))$ for each $l \in \mathbb{N}$ and Proposition 3.17, this implies that $\{C_l\}_{l\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. Since $\{B_l\}_{l\in\mathbb{N}}$ is a topological sub-basis of (X, ρ) and K_i is compact for each $i \in I$, we have $X \smallsetminus K_i = \bigcup\{B_k : K_i \subseteq C_k\}$ for each $i \in I$. Therefore, by Propositions 3.22 and 3.8, $\{X \smallsetminus K_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets.

(iii) For all $i \in I$, sequence $\{m_n\}_{n=1}^p$ of integers, and sequence $\{q_n\}_{n=1}^p$ of positive rational numbers, we obtain that $K_i \smallsetminus U_i \subseteq \bigcup_{n=1}^p B(s_{m_n}, q_n)$ if and only if $K_i \subseteq U_i \cup \big(\bigcup_{n=1}^p B(s_{m_n}, q_n)\big)$. Therefore, it follows from Propositions 3.10 and 3.22 that $\{K_i \smallsetminus U_i\}_{i\in\mathbb{N}}$ is a sequence of uniformly recursively compact sets.

(iv) Let $\{q_k\}_{k\in\mathbb{N}}$ be an effective enumeration of $\mathbb{Q}$. Since $\{f_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable functions with respect to $\{K_i\}_{i\in I}$, by Definition 3.18, $\big\{f_i^{-1}((q_k, +\infty))\big\}_{(i,k)\in I\times\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets. Note that for each $i \in I$, $\inf_{x\in K_i} f_i(x) = \sup\big\{q_k \in \mathbb{Q} : K_i \subseteq f_i^{-1}((q_k, +\infty))\big\}$. Then by Proposition 3.22 and Definition 3.3, we obtain that $\big\{\inf_{x\in K_i} f_i(x)\big\}_{i\in I}$ is a sequence of uniformly lower semi-computable real numbers.

Now we assume that $\{f_i\}_{i\in I}$ is a sequence of uniformly upper semi-computable functions. Then by Definition 3.18, $\{-f_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable functions. By the previous result, $\big\{\inf_{x\in K_i}(-f_i(x))\big\}_{i\in I}$ is a sequence of uniformly lower semi-computable real numbers. Note that $\inf_{x\in K_i}(-f_i(x)) = -\sup_{x\in K_i} f_i(x)$ for each $i \in I$. Then by Definition 3.3, $\big\{\sup_{x\in K_i} f_i(x)\big\}_{i\in I}$ is a sequence of uniformly upper semi-computable real numbers.

(v) Denote by $\{U_l\}_{l\in\mathbb{N}}$ by an effective enumeration of $\big\{\bigcup_{n=1}^p B(s_{m_n}, q_n) : p \in \mathbb{N},\, m_n \in \mathbb{N}, q_n \in \mathbb{Q}^+,\, 1 \leqslant n \leqslant p\big\}$. Since $\{T_i\}_{i\in I}$ is a sequence of uniformly computable functions with respect to $\{K_i\}_{i\in I}$, by Proposition 3.17, there exists a sequence $\{V_{i,l}\}_{(i,l)\in I\times\mathbb{N}}$ of uniformly lower semi-computable open sets such that $T_i^{-1}(U_l) \cap K_i = V_{i,l} \cap K_i$ for all $i \in I$ and $l \in \mathbb{N}$. Thus, we obtain that for all $i \in I$ and $l \in \mathbb{N}$, $T_i(K_i) \subseteq U_l$ is equivalent to $K_i \subseteq V_{i,l}$. Hence, by Definition 3.21 and Proposition 3.22, $\{T_i(K_i)\}_{i\in\mathbb{N}}$ is a sequence of uniformly recursively compact sets. □

Next, we investigate whether the property of uniform computability for recursively compact sets is preserved under the union and intersection.

Proposition 3.24. *Let $(X, \rho, \mathcal{S})$ be a computable metric space in which X is recursively compact. Suppose that H and L are two nonempty recursively enumerable sets with $L \subseteq I \times H$, and $\{K_{i,h}\}_{(i,h)\in L}$ is a sequence of uniformly recursively compact sets. Denote $L_h \coloneqq \{(i, h) \in L : i \in I\}$ for each $h \in H$. Then the following statements are true:*

- (i) *$\{\bigcap\{K_{i,h} : (i, h) \in L_h\}\}_{h\in H}$ is uniformly recursively compact.*
- (ii) *If the function $F\colon H \to \mathbb{N}$ defined by $F(h) \coloneqq \operatorname{card} L_h$ for $h \in H$ is recursive, then $\{\bigcup\{K_{i,h} : (i, h) \in L_h\}\}_{h\in H}$ is uniformly recursively compact.*

Proposition 3.24 (i) follows immediately from Proposition 3.10 and Proposition 3.23 (ii) and (iii). Moreover, Proposition 3.24 (ii) follows from Definition 3.21. As a corollary of Proposition 3.24 (ii), we obtain the following result.

Corollary 3.25. *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Assume that X is recursively compact, $T\colon X \to X$ is a computable function, and $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open. Then $\{V_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ is uniformly lower semi-computable open, where $V_{n,i}$ is defined inductively by setting $V_{1,i} \coloneqq U_i$ and $V_{n+1,i} \coloneqq T^{-1}(V_{n,i}) \cap U_i$ for each $n \in \mathbb{N}$ and each $i \in I$.*

Proof. Since T is a computable function, by Definition 3.2, we obtain that $\{T^n\}_{n\in\mathbb{N}_0}$ is a sequence of uniformly computable functions. Then by Proposition 3.17, $\{T^{-n}(U_i)\}_{(n,i)\in\mathbb{N}_0\times I}$ is a sequence of uniformly lower semi-computable open sets. Hence, since X is recursively compact, by Proposition 3.23 (iii), $\{X \setminus T^{-n}(U_i)\}_{(n,i)\in\mathbb{N}_0\times I}$ is a sequence of uniformly recursively compact sets.

Define $L \subseteq \mathbb{N}_0 \times \mathbb{N} \times I$ by $L \coloneqq \{(m,n,i) \in \mathbb{N}_0 \times \mathbb{N} \times I : m < n\}$. Then L is a recursively enumerable set and $\{X \setminus T^{-m}(U_i)\}_{(m,n,i)\in L}$ is uniformly recursively compact. Note that $\operatorname{card}\{m \in \mathbb{N}_0 : (m,n,i) \in L\} = n$ for all $n \in \mathbb{N}$ and $i \in I$. Then the function $F \colon \mathbb{N} \times I \to \mathbb{N}_0$ given by $F(n,i) \coloneqq \operatorname{card}\{m \in \mathbb{N}_0 : (m,n,i) \in L\}$ is a recursive function. Thus, by Proposition 3.24 (ii), we obtain that $\{\bigcup\{X \setminus T^{-m}(U_i) : m \in \mathbb{N}_0,\, (m,n,i) \in L\}\}_{(n,i)\in\mathbb{N}\times I}$ is a sequence of uniformly recursively compact sets. Hence, since X is a recursively compact set, by Proposition 3.23 (ii), $\{X \setminus \bigcup\{X \setminus T^{-m}(U_i) : m \in \mathbb{N}_0,\, (m,n,i) \in L\}\}_{(n,i)\in\mathbb{N}\times I} = \{\bigcap\{T^{-m}(U_i) : m \in \mathbb{N}_0,\, (m,n,i) \in L\}\}_{(n,i)\in\mathbb{N}\times I}$ is uniformly lower semi-computable open. Since $V_{1,i} = U_i$ and $V_{n+1,i} = T^{-1}(V_{n,i}) \cap U_i$ for all $n \in \mathbb{N}$ and $i \in I$, it follows by induction that $V_{n+1,i} = \bigcap_{k=0}^{n} T^{-k}(U_i)$ for each $n \in \mathbb{N}_0$. Therefore, $\{V_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ is uniformly lower semi-computable open. □

Moreover, given the recursive compactness of X, the computability of functions is preserved under a finite number of operations among additions and multiplications. We summarize this property in the following result (cf. [We00, Corollary 4.3.4]).

Proposition 3.26. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space in which X is recursively compact, and H be a nonempty recursively enumerable set. Assume that $\{f_i\}_{i\in I}$ (resp. $\{g_h\}_{h\in H}$) is a sequence of uniformly computable functions $f_i \colon X \to \mathbb{R}$ (resp. $g_h \colon X \to \mathbb{R}$). Then $\{f_i + g_h\}_{(i,h)\in I\times H}$, $\{f_i \cdot g_h\}_{(i,h)\in I\times H}$ are two sequences of uniformly computable functions.*

Next, we recall the definition of recursively precompact metric space.

Definition 3.27 (Recursively precompact metric space). Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$. Then $(X,\, \rho,\, \mathcal{S})$ is called *recursively precompact* if there exists an algorithm that, for each $n \in \mathbb{N}$, on input n, outputs a finite subset $\{r_i : 1 \leqslant i \leqslant m\}$ of $\mathbb{N}$ such that $X = \bigcup_{i=1}^{m} B(s_{r_i}, 2^{-n})$.

Finally, we record [GHR11, Proposition 4] which characterizes complete recursively precompact metric spaces.

Proposition 3.28. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space. Then X is recursively compact if and only if (X, ρ) is complete and $(X,\, \rho,\, \mathcal{S})$ is recursively precompact.*

3.6. **Computability of probability measures.** Building upon the theory of computable functions and recursively compact sets, we now discuss the computability of probability measures. We begin by reviewing the computable structure on the measure space $\mathcal{P}(X)$ introduced in [HR09, Section 4].

Proposition 3.29. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space in which X is recursively compact. Then the following statements are true:*

(i) *Let $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$. Then there exists an enumeration $\mathcal{Q}_{\mathcal{S}} = \{\nu_k\}_{k\in\mathbb{N}}$ of the set of Borel probability measures that are supported on finitely many points in $\{s_n : n \in \mathbb{N}\}$ and assign rational values to them such that there exists an algorithm that, for each $k \in \mathbb{N}$, upon input k, outputs a sequence $\{n_l\}_{l=1}^{p}$ of integers and a sequence $\{q_l\}_{l=1}^{p}$ of positive rational numbers satisfying that $\sum_{l=1}^{p} q_l = 1$ and $\nu_k = \sum_{l=1}^{p} q_l \delta_{s_{n_l}}$.*

(ii) *$(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_{\mathcal{S}})$ is also a computable metric space in which $\mathcal{P}(X)$ is recursively compact, where W_ρ is the Wasserstein–Kantorovich metric on $\mathcal{P}(X)$ (see (2.2)).*

Proof. (i) This follows from the fact that $\mathbb{N}^*$ is a recursively enumerable set.

(ii) Since X is recursively compact, by Definition 3.21 and Proposition 3.28, (X, ρ) is a bounded and complete metric space and $(X,\, \rho,\, \mathcal{S})$ is a recursively precompact computable metric space. Thus, by [HR09, Proposition 4.1.3], $(\mathcal{P}(X), W_\rho)$ is complete and $(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_\mathcal{S})$ is also a computable metric space.

Since $(X,\, \rho,\, \mathcal{S})$ is a recursively precompact computable metric space, by [HR09, Lemma 2.12], $(\mathcal{P}(X), W_\rho, \mathcal{Q}_\mathcal{S})$ is also a recursively precompact computable metric space. Thus, combined with the completeness of $(\mathcal{P}, W_\rho)$ and Proposition 3.28, this implies that $\mathcal{P}(X)$ is recursively compact in $(\mathcal{P}(X), W_\rho, \mathcal{Q}_\mathcal{S})$. □

Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space in which X is recursively compact. We endow the measure space $\mathcal{P}(X)$ with the computable structure $(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_\mathcal{S})$ given by Proposition 3.29.

The computability of measures is then defined via Definition 3.5. Specifically, a sequence $\{\mu_i\}_{i\in I}$ of measures in $\mathcal{P}(X)$ is *a sequence of uniformly computable measures* if it is uniformly computable in $(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_\mathcal{S})$, and a single measure $\mu \in \mathcal{P}(X)$ is *a computable measure* if the corresponding constant sequence consisting of μ is uniformly computable. As a remark, by [HR09, Theorem 4.1.1]the above computability notion is equivalent to the one defined in [HR09, Definition 4.1.2].

We now recall a key result on the computability of the integration function (cf. [HR09, Corollary 4.3.2]).

Proposition 3.30. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space. Assume that X is recursively compact in $(X,\, \rho,\, \mathcal{S})$, and that $\{f_i\}_{i\in I}$ is a sequence of uniformly computable functions $f_i\colon X \to \mathbb{R}$. Then the sequence $\{\mathcal{I}_i\}_{i\in I}$ of functions $\mathcal{I}_i\colon \mathcal{P}(X) \to \mathbb{R}$ defined by $\mathcal{I}_i(\mu) \coloneqq \langle \mu, f_i\rangle$ for $\mu \in \mathcal{P}(X)$ is a sequence of uniformly computable functions.*

Proof. Suppose $\mathcal{S} = \{s_k\}_{k\in\mathbb{N}}$. Since $\{f_i\}_{i\in I}$ is a sequence of uniformly computable functions, by Proposition 3.23 (iv) and Corollary 3.20, $\{\sup_{x\in X} f_i(x)\}_{i\in I}$ is a sequence of uniformly upper semi-computable real numbers. Note that $\{s_k\}_{k\in\mathbb{N}}$ is a sequence of uniformly computable points. Then it follows from Definitions 3.5 and 3.14 that $\{f_i(s_k)\}_{(i,k)\in I\times\mathbb{N}}$ is a sequence of uniformly computable real numbers. Hence, it is not difficult to derive from Definition 3.3 that $\{\sup_{k\in\mathbb{N}} f_i(s_k)\}_{i\in I}$ is a sequence of uniformly lower semi-computable real numbers. Since $\{s_k : k \in \mathbb{N}\}$ is dense in X and f_i is a continuous function on X, we have $\sup_{x\in X} f_i(x) = \sup_{k\in\mathbb{N}} f_i(s_k)$ for each $i \in I$. Hence, by Corollary 3.20, we obtain that $\{\sup_{x\in X} f_i(x)\}_{i\in I}$ is a sequence of uniformly computable real numbers. Thus, by [HR09, Corollary 4.3.2], it follows from the uniform computability of $\{f_i\}_{i\in I}$ that the function $\mathcal{I}\colon \mathcal{P}(X) \times I \to \mathbb{R}$ defined by $\mathcal{I}(\mu, i) \coloneqq \mathcal{I}_i(\mu)$ is computable. Therefore, by Definition 3.14, we obtain that $\{\mathcal{I}_i\}_{i\in I}$ is a sequence of uniformly computable functions. □

As an immediate corollary of Definition 3.18 and Proposition 3.30, we have the following result.

Corollary 3.31. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space. Assume that X is recursively compact in $(X,\, \rho,\, \mathcal{S})$, and that $\{f_i\}_{i\in I}$ is a sequence of uniformly upper (resp. lower) semi-computable functions $f_i\colon X \to \mathbb{R}$. Then the sequence $\{\mathcal{I}_i\}_{i\in I}$ of functions $\mathcal{I}_i\colon \mathcal{P}(X) \to \mathbb{R}$ given by $\mathcal{I}_i(\mu) \coloneqq \langle \mu, f_i\rangle$ is a sequence of uniformly upper (resp. lower) semi-computable functions.*

Finally, we consider a family of computable functions as follows.

Definition 3.32. Let (X, ρ) be a metric space. Consider arbitrary $r \in \mathbb{R}$, $\epsilon > 0$, and $u \in X$. Then the function $g_{u,r,\epsilon}\colon X \to \mathbb{R}$ given by

$$g_{u,r,\epsilon}(x) \coloneqq \big(1 - (1/\epsilon)(\rho(x,u) - r)^+\big)^+, \quad \text{for } x \in X, \tag{3.1}$$

is called a *hat function*.

If $r > 0$, then $g_{u,r,\epsilon}$ is a $(1/\epsilon)$-Lipschitz function that equals 1 in the closed ball $\overline{B}(u, r)$ and 0 outside the ball $B(u, r + \epsilon)$, and lies strictly between 0 and 1 in the annulus $B(u, r + \epsilon) \smallsetminus \overline{B}(u, r)$.

The following result indicates that the characteristic function of a lower semi-computable open set is a lower semi-computable function.

Proposition 3.33. *Let $(X, \rho, \mathcal{S})$ be a computable metric space. Assume that $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets. Then there exists a sequence $\{h_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ of uniformly computable functions $h_{n,i} \colon X \to \mathbb{R}$ such that for each $i \in I$, the following properties are satisfied:*

(i) *For each $x \in X$, $\{h_{n,i}(x)\}_{n\in\mathbb{N}}$ is non-decreasing and converges to $\mathbb{1}_{U_i}(x)$ as $n \to +\infty$.*
(ii) *For each $n \in \mathbb{N}$, $h_{n,i}(x) \geqslant 0$ for each $x \in X$ and $h_{n,i}(x) = 0$ for each $x \notin U_i$.*

Proof. Suppose $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$. Let $\{q_n\}_{n\in\mathbb{N}}$ (resp. $\{D_n\}_{n\in\mathbb{N}}$) be an effective enumeration of $\mathbb{Q}$ (resp. the extended ideal balls in $(X, \rho, \mathcal{S})$). Then there exist three recursive functions $f \colon \mathbb{N} \to \mathbb{N}$, $u \colon \mathbb{N} \to \mathbb{N}$, and $v \colon \mathbb{N} \to \mathbb{N}$ such that $D_n = B\bigl(s_{f(n)}, \frac{u(n)-1}{v(n)}\bigr)$ for each $n \in \mathbb{N}$. Since $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open, by Definition 3.7, there exists a recursive function $l \colon \mathbb{N} \times I \to \mathbb{N}$ such that $U_i = \bigcup_{n\in\mathbb{N}} D_{l(n,i)}$ for each $i \in I$. Writing $w_{m,i,n} \coloneqq \frac{u(l(m,i))-1}{v(l(m,i))} - \frac{1}{n}$, we define $h_{n,i} \colon X \to \mathbb{R}$ by

$$h_{n,i}(x) \coloneqq \max\bigl\{g_{s_{f(l(m,i))}, w_{m,i,n}, 1/n}(x) : m \in \mathbb{N} \cap [1, n]\bigr\} \quad \text{for all } n \in \mathbb{N},\ i \in I, \text{ and } x \in X.$$

By (3.1) and Definition 3.2, it follows that $\{h_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ is a sequence of uniformly computable functions which satisfies properties (i) and (ii). □

The following result is also useful to describe the computability of subsets of $\mathcal{P}(X)$.

Proposition 3.34. *Let $(X, \rho, \mathcal{S})$ be a computable metric space in which X is recursively compact. Assume that H and L are two nonempty recursively enumerable sets with $L \subseteq I \times H$, $\{U_{i,h}\}_{(i,h)\in L}$ is a sequence of uniformly lower semi-computable open sets in $(X, \rho, \mathcal{S})$, and $\{r_{i,h}\}_{(i,h)\in L}$ is a sequence of uniformly computable real numbers. Define, for each $i \in I$, $L_i \coloneqq \{(i, h) \in L : h \in H\}$ and $\mathcal{K}_i \coloneqq \{\mu \in \mathcal{P}(X) : \mu(U_{i,h}) \leqslant r_{i,h}$ for each $(i, h) \in L_i\}$. Then $\{\mathcal{K}_i\}_{i\in I}$ is uniformly recursively compact in $(\mathcal{P}(X), W_\rho, \mathcal{Q}_{\mathcal{S}})$.*

Proof. Define $\{\mathcal{I}_{i,h}\}_{(i,h)\in L}$ by $\mathcal{I}_{i,h}(\mu) \coloneqq \mu(U_{i,h})$ for all (i, h) and $\mu \in \mathcal{P}(X)$. Since $\{U_{i,h}\}_{(i,h)\in L}$ is uniformly lower semi-computable open, by Proposition 3.33 and Definition 3.18, $\{\mathbb{1}_{U_{i,h}}\}_{(i,h)\in L}$ is a sequence of uniformly lower semi-computable functions. Hence, by Corollary 3.31, the sequence $\{\mathcal{I}_{i,h}\}_{(i,h)\in L}$ is a sequence of uniformly lower semi-computable functions. Since $\{r_{i,h}\}_{(i,h)\in L}$ is a sequence of uniformly computable real numbers, $\{(r_{i,h}, +\infty)\}_{(i,h)\in L}$ is a sequence of uniformly lower semi-computable open sets. Combined with the uniform lower semi-computability of $\{\mathcal{I}_{i,h}\}_{(i,h)\in L}$, by Proposition 3.19, this implies that the sequence $\{\mathcal{U}_{i,h}\}_{(i,h)\in L}$ defined by $\mathcal{U}_{i,h} \coloneqq \{\mu \in \mathcal{P}(X) : \mathcal{I}_{i,h}(\mu) > r_{i,h}\}$ for each $(i, h) \in L$ is a sequence of uniformly lower semi-computable open sets. Hence, by Proposition 3.10, $\bigl\{\bigcup_{(i,h)\in L_i} \mathcal{U}_{i,h}\bigr\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets. Note that by the definition of $\{\mathcal{K}_i\}_{i\in I}$, $\mathcal{K}_i = \mathcal{P}(X) \smallsetminus \bigl(\bigcup_{(i,h)\in L_i} \mathcal{U}_{i,h}\bigr)$ for each $i \in I$. Since X is recursively compact, by Proposition 3.29, $\mathcal{P}(X)$ is recursively compact. Therefore, by Proposition 3.23 (iii), $\{\mathcal{K}_i\}_{i\in I}$ is uniformly recursively compact. □

The following proposition follows immediately from the Stone–Weierstrass theorem and the dominated convergence theorem.

Proposition 3.35. *Let $(X, \rho, \mathcal{S})$ be a computable metric space, and X be a recursively compact set in $(X, \rho, \mathcal{S})$. Then there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of uniformly computable functions $\tau_n \colon X \to \mathbb{R}$ such that $\{\tau_n : n \in \mathbb{N}\}$ is dense in $C(X)$. Hence, $\mu, \nu \in \mathcal{M}(X)$, $\mu(A) \geqslant \nu(A)$ for each $A \in \mathcal{B}(X)$ if and only if $\langle \mu, \tau_n^+ \rangle \geqslant \langle \nu, \tau_n^+ \rangle$ for each $n \in \mathbb{N}$. by (3.2),* *Proof.* First, we assume that $\{\tau_n : n \in \mathbb{N}\}$ is dense in $C(X)$ and consider two arbitrary measure μ, $\nu \in \mathcal{M}(X)$. Now we demonstrate that $\mu(A) \geqslant \nu(A)$ for each $A \in \mathcal{B}(X)$ if and only if $\langle \mu, \tau_n^+ \rangle \geqslant \langle \nu, \tau_n^+ \rangle$ for each $n \in \mathbb{N}$. Hence, since $\{\tau_n : n \in \mathbb{N}\}$ is dense in $C(X)$, by the Dominated Convergence Theorem, $\langle \mu, \tau_n^+ \rangle \geqslant \langle \nu, \tau_n^+ \rangle$ for each $n \in \mathbb{N}$ if and only if $\langle \mu, \tau^+ \rangle \geqslant \langle \mu, \tau^+ \rangle$ for each $\tau \in C(X)$. Note that $\mathcal{M}(X)$ is the dual space of $C(X)$. Then $\langle \mu, \tau^+ \rangle \geqslant \langle \mu, \tau^+ \rangle$ for each $\tau \in C(X)$ if and only if $\mu(A) \geqslant \nu(A)$ for each $A \in \mathcal{B}(X)$.

Now we construct a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of uniformly computable functions $\tau_n \colon X \to \mathbb{R}$ such that $\{\tau_n : n \in \mathbb{N}\}$ is dense in $C(X)$. Let $\mathcal{F}_0(\mathcal{S}) \coloneqq \{g_{s_m,p/q,1/n} : p,\, q,\, m,\, n \in \mathbb{N}\}$ and $\mathfrak{E}(\mathcal{S})$ be the smallest (in the sense of inclusion) set of functions containing $\mathcal{F}_0(\mathcal{S})$ and the constant function $\mathbb{1}_X$, closed under a finite number of operators from the following list: additions, multiplications, and scalar multiplications with rational numbers. By Stone–Weierstrass theorem (see e.g., [Fo99, Theorem 4.45]), it immediately follows from the above definition of $\mathfrak{E}(\mathcal{S})$ that $\mathfrak{E}(\mathcal{S})$ is dense in $C(X)$.

Finally, we give an effective enumeration of $\mathfrak{E}(\mathcal{S})$. Indeed, since $\mathbb{N}^*$ is a recursively enumerable set, there exists an enumeration $\{\tau_n\}_{n\in\mathbb{N}}$ of $\mathfrak{E}(\mathcal{S})$ and a corresponding algorithm that, for each $n \in \mathbb{N}$, on input n, outputs the expression of the function τ_n. Therefore, it follows from Definitions 3.32 and 3.14 that $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly computable functions. □

For a compact metric space (X,ρ), a Borel-measurable transformation $T \colon X \to X$, we say that $A \subseteq X$ is *admissible* (for T) if A, $T(A) \in \mathcal{B}(X)$ and $T|_A$ is injective. Given a Borel subset $Y \subseteq X$, define

$$\mathcal{M}(X,T;Y) \coloneqq \{\mu \in \mathcal{P}(X) : \mu\big(T^{-1}(A) \cap Y\big) \leqslant \mu(A) \text{ for each Borel } A \subseteq X\}. \tag{3.2}$$

The following result indicates the recursive compactness of the set $\mathcal{M}(X,T;Y)$ (cf. [BHLZ25, Lemma 4.12]).

Proposition 3.36. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space in which X is recursively compact, and $\{U_i\}_{i\in I}$ be a sequence of uniformly lower semi-computable open sets. Assume that $\{T_i\}_{i\in I}$ is a sequence of uniformly computable functions $T_i \colon X \to X$ with respect to $\{U_i\}_{i\in I}$. Then $\{\mathcal{M}(X,T_i;U_i)\}_{i\in I}$ is uniformly recursively compact in $(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_{\mathcal{S}})$. In particular, if $\{T_i\}_{i\in I}$ is a sequence of uniformly computable functions, then $\{\mathcal{M}(X,T_i)\}_{i\in I}$ is uniformly recursively compact.*

Proof. Since $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open, by Proposition 3.33, $\{\mathbb{1}_{U_i}\}_{i\in I}$ is a sequence of uniformly lower semi-computable functions. By Proposition 3.35, there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of uniformly computable functions $\tau_n \colon X \to \mathbb{R}$ such that $\{\tau_n : n \in \mathbb{N}\}$ is dense in $C(X)$. Hence, by (3.2), we obtain that

$$\mathcal{M}(X,T_i;U_i) = \bigcap_{n\in\mathbb{N}} \{\mu \in \mathcal{P}(X) : \langle \mu, \big(\tau_n^+ \circ T_i\big) \cdot \mathbb{1}_{U_i} \rangle \leqslant \langle \mu, \tau_n^+ \rangle\} \quad \text{for each } i \in I. \tag{3.3}$$

Since $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly computable functions, and $\{T_i\}_{i\in I}$ is a sequence of uniformly computable functions with respect to $\{U_i\}_{i\in I}$, by Propositions 3.17, we obtain that $\{\tau_n^+ \circ T_i\}_{(n,i)\in\mathbb{N}\times I}$ is a sequence of uniformly computable functions with respect to $\{U_i\}_{i\in I}$. Hence, since τ_n^+ is nonnegative function for each $n \in \mathbb{N}$, it follows from uniformly lower semi-computability of $\{U_i\}_{i\in I}$ and $\{\mathbb{1}_{U_i}\}_{i\in I}$ and Definition 3.18 that $\{\big(\tau_n^+ \circ T_i\big) \cdot \mathbb{1}_{U_i}\}_{(n,i)\in\mathbb{N}\times I}$ is a sequence of uniformly computable functions. Thus, since $\mathbb{R}^+$ is a lower semi-computable open set, by Proposition 3.30, the sequence $\{V_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ is a sequence of uniformly lower semi-computable open sets, where $V_{n,i} \coloneqq \{\mu \in \mathcal{P}(X) : \langle \mu, \big(\tau_n^+ \circ T_i\big) \cdot \mathbb{1}_{U_i} \rangle > \langle \mu, \tau_n^+ \rangle\}$ for all $n \in \mathbb{N}$ and $i \in I$. Thus, by Proposition 3.10, $\{\bigcup_{n\in\mathbb{N}} V_{n,i}\}_{i\in I}$ is uniformly lower semi-computable open. Since X is recursively compact, by Proposition 3.29, $\mathcal{P}(X)$ is recursively compact. Therefore, by Proposition 3.23 (iii) and (3.3), we obtain that $\{\mathcal{M}(X, T_i; U_i)\}_{i\in I}$ is uniformly recursively compact. □

3.7. Computability over the Riemann sphere.

Proposition 3.37. *Let $\mathcal{S}\bigl(\widehat{\mathbb{C}}\bigr) \coloneqq \{s_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}\bigl(\widehat{\mathbb{C}}\bigr) \coloneqq \{a + b\mathbf{i} : a,\, b \in \mathbb{Q}\}$ such that there exists an algorithm that for each $n \in \mathbb{N}$, upon input n, outputs $p_1,\, q_1,\, r_1,\, p_2,\, q_2,\, r_2 \in \mathbb{N}$ with $s_n = (-1)^{r_1}\frac{p_1}{q_1} + (-1)^{r_2}\frac{p_2}{q_2}\mathbf{i}$. Then $\bigl(\widehat{\mathbb{C}},\, \sigma,\, \mathcal{S}\bigl(\widehat{\mathbb{C}}\bigr)\bigr)$ is a computable metric space in which $\widehat{\mathbb{C}}$ is recursively compact, where σ is the chordal metric on $\widehat{\mathbb{C}}$.*

Proof. Since $\mathbb{Q}\bigl(\widehat{\mathbb{C}}\bigr)$ is dense in $\bigl(\widehat{\mathbb{C}}, \sigma\bigr)$. Then Definition 3.4 (i) holds. By the definition of $\mathcal{S}\bigl(\widehat{\mathbb{C}}\bigr)$, Definition 3.4 (ii) holds. Moreover, by the definition of the chordal metric σ, $\{\sigma(s_m, s_n)\}_{(m,n)\in\mathbb{N}^2}$ is a sequence of uniformly computable real numbers, hence, Definition 3.4 (iii) holds. Thus, $\bigl(\widehat{\mathbb{C}},\, \sigma,\, \mathcal{S}\bigl(\widehat{\mathbb{C}}\bigr)\bigr)$ is a computable metric space. By Definition 3.27, $\bigl(\widehat{\mathbb{C}},\, \sigma,\, \mathcal{S}\bigl(\widehat{\mathbb{C}}\bigr)\bigr)$ is recursively precompact. Combined with the completeness of (X, σ), by Proposition 3.28, this implies that $\widehat{\mathbb{C}}$ is a recursively compact set. □

Proposition 3.38. *There exists an algorithm that satisfies the following property:*

For each $m \in \mathbb{N}$, each $n \in \mathbb{N}$, and each complex polynomial p of degree m, this algorithm outputs a sequence $\{q_i\}_{i=1}^m$ of integers satisfying that if $x_1, x_2, \dots, x_m$ are all the zeros of the map p (counting with multiplicity), then there exists a permutation σ on $\{1, 2, \dots, m\}$ such that $\sigma\bigl(u_{q_{\sigma(i)}}, x_i\bigr) < 2^{-n}$ for each integer $1 \leqslant i \leqslant m$, where $\bigl\{u_j\bigr\}_{j\in\mathbb{N}}$ is the effective enumeration of the set $\mathbb{Q}\bigl(\widehat{\mathbb{C}}\bigr)$, after we input the following data in this algorithm:

- (i) *an algorithm $\mathcal{A}_p$ computing all the coefficients of the polynomial p,*
- (ii) *the integer n.*

Proof. Let $\{s_i\}_{i\in\mathbb{N}}$ be an effective enumeration of the set $\{a + b\mathbf{i} : a,\, b \in \mathbb{Q}\}$. Now we design an algorithm $M(\,\cdot\,,\cdot\,)$ satisfying the following property:

For each polynomial Q, there exists a zero z_0 of Q satisfying that for each $m \in \mathbb{N}$, $M\bigl(\mathcal{A}_Q, m\bigr)$ outputs a point $l_m \in \mathbb{Q}\bigl(\widehat{\mathbb{C}}\bigr)$ with $\sigma(l_m, z_0) < 2^{-m}$ after we input an algorithm $\mathcal{A}_Q$ computing all the coefficients of the polynomial Q and the integer m.

First, we use the algorithm $\mathcal{A}_Q$ to compute the sequence $\{Q'(s_i)\}$ and select a subsequence $\{\tilde{s}_i\}_{i\in\mathbb{N}}$ of $\{s_i\}_{i\in\mathbb{N}}$ of all the ideal points $\tilde{s}_i$ with $Q'(\tilde{s}_i) \neq 0$. Then we define two sequences $\{\gamma(Q, i)\}_{i\in\mathbb{N}}$ and $\{\beta(Q, i)\}_{i\in\mathbb{N}}$ by

$$\gamma(Q, i) \coloneqq \sup_{k\geqslant 2}\left|\frac{Q^{(k)}(\tilde{s}_i)}{k!Q'(\tilde{s}_i)}\right|^{\frac{1}{k-1}} \quad \text{and} \quad \beta(Q, i) \coloneqq \left|\frac{Q(\tilde{s}_i)}{Q'(\tilde{s}_i)}\right|. \tag{3.4}$$

Since there exist finitely many roots for the rational map Q' and $\{s_i\}_{i\in\mathbb{N}}$ is dense in $\mathbb{C}$, $\{\tilde{s}_i\}_{i\in\mathbb{N}}$ is also dense in $\mathbb{C}$. Combining with the fact that $\beta(Q, \xi) = 0$ for each root $\xi \in \mathbb{C}$ of Q, we can enumerate the sequence $\{\tilde{s}_i\}_{i\in\mathbb{N}}$ and find $i_0 \in \mathbb{N}$ with $\alpha(Q, i_0) \coloneqq \beta(Q, i_0)\gamma(Q, i_0) < \alpha_0$ (here we can select $\alpha_0 \coloneqq 0.03$, see Remark 6 of [BCSS98, Section 8.2]). Next, compute an integer k_m with

$$k_m > \log_2(m + 4 + \log_2(\beta(Q, i_0))). \tag{3.5}$$

Hence, by Theorem 2 of [BCSS98, Section 8.2], there exists a zero $z_0 \in \mathbb{C}$ of Q satisfying that

$$\bigl|N_Q^t(\tilde{s}_{i_0}) - z_0\bigr| \leqslant \frac{|\tilde{s}_{i_0} - z_0|}{2^{2^t-1}} \leqslant \frac{2\beta(Q, i_0)}{2^{2^t-1}} \quad \text{for each } t \in \mathbb{N}.$$

Here $N_Q(z) \coloneqq z - \frac{Q(z)}{Q'(z)}$ for each $z \in \mathbb{C}$. Combining with (3.5), this implies that

$$\bigl|N_Q^{k_m}(\tilde{s}_{i_0}) - z_0\bigr| \leqslant \frac{2\beta(Q, i_0)}{2^{2^{k_m}-1}} < \frac{2\beta(Q, i_0)}{2^{m+3+\log_2(\beta(Q,i_0))}} = \frac{1}{2^{m+2}}. \tag{3.6}$$

Finally, we use the algorithm $\mathcal{A}_Q$ to compute and output a point $l_m \in \mathbb{Q}(\widehat{\mathbb{C}})$ with $\bigl|l_m - N_Q^{k_m}(\tilde{s}_{i_0})\bigr| < 2^{-m-2}$. It follows immediately from the definition of the chordal metric σ on $\widehat{\mathbb{C}}$ (see Section 2) that $\sigma(z, w) \leqslant 2|z - w|$ for each pair of z, $w \in \mathbb{C}$. Hence, by (3.6),

$$\sigma(l_m, z_0) \leqslant 2|l_m - z_0| \leqslant 2\bigl(\bigl|l_m - N_Q^{k_m}(\tilde{s}_{i_0})\bigr| + \bigl|N_Q^{k_m}(\tilde{s}_{i_0}) - z_0\bigr|\bigr) < 2^{-m}.$$

So far we have designed the algorithm $M(\cdot, \cdot)$.

Next, we come back to the proof of the original statement. Fix an integer n and a complex polynomial p of degree n. First, we can use the algorithm $M\bigl(\mathcal{A}_p, \cdot\bigr)$ to compute a zero of the polynomial p, say z_0. Then we consider the map $\overline{p}(z) := \frac{p(z)}{z - z_0}$. Since $p(z_0) = 0$, $\overline{p}$ is a polynomial of degree $n - 1$. Now we claim that we can compute all the coefficients of the polynomial $\overline{p}$ from the point z_0 and all the coefficients of the polynomial p. Indeed, if $p(z) = \sum_{i=0}^{n} a_i z^i$ and $\overline{p}(z) = \sum_{i=0}^{n-1} b_i z^i$, then it is not hard to see that $b_i = a_{i+1} + z_0 b_{i+1}$ for each integer $0 \leqslant i \leqslant n - 1$, where b_n is set to be 0. Hence, we obtain an algorithm $\mathcal{A}_{\overline{p}}$ computing all the coefficients of $\overline{p}$. Then we can use the algorithm $M\bigl(\mathcal{A}_{\overline{p}}, \cdot\bigr)$ to compute a zero of the polynomial $\overline{p}$, i.e., a new zero of the polynomial p. Therefore, we can compute all the zeros of p (counting with multiplicity) recursively. □

4. Ergodic theory

We review basic concepts from ergodic theory. For more detailed discussions, we refer the reader to [Wa82, Section 4].

Let $(X, \mathcal{B}, \mu)$ be a probability space. A *partition* $\xi = \{A_h : h \in H\}$ of $(X, \mathcal{B}, \mu)$ is a disjoint collection of elements of $\mathcal{B}$ whose union is X, where H is a countable index set. For each pair of partitions $\xi = \{A_h : h \in H\}$ and $\eta = \{B_l : l \in L\}$ of X, their *join* is the partition $\xi \vee \eta := \{A_h \cap B_l : h \in H, l \in L\}$.

Assume that $T\colon X \to X$ is a measure-preserving transformation of $(X, \mathcal{B}, \mu)$. Consider a partition $\xi = \{A_h : h \in H\}$ of X. For each $n \in \mathbb{N}$, $T^{-n}(\xi)$ denotes the partition $\bigl\{T^{-1}(A_h) : h \in H\bigr\}$, and ξ_T^n denotes the join $\xi \vee T^{-1}(\xi) \vee \cdots \vee T^{-(n-1)}(\xi)$. The *entropy* of ξ is $H_\mu(\xi) := -\sum_{h\in H} \mu(A_h) \log(\mu(A_h)) \in [0, +\infty]$, where $0 \log 0$ is defined to be zero. One can show that if $H_\mu(\xi) < +\infty$, then $\lim_{n\to+\infty} H_\mu(\xi_T^n)/n$ exists (see e.g. [Wa82, Chapter 4]). We denote this limit by $h_\mu(T, \xi)$ and call it the *measure-theoretic entropy of T relative to ξ*. The *measure-theoretic entropy* of T for μ is defined as

$$h_\mu(T) := \sup\{h_\mu(T, \xi) : \xi \text{ is a partition of } X \text{ with } H_\mu(\xi) < +\infty\}. \tag{4.1}$$

We now introduce thermodynamic formalism, a particular branch of ergodic theory. The main objects of study are the topological pressure and equilibrium states (see e.g. [PU10, Wa82]; for the general Borel-measurable setting used in Approach II, see e.g. [IT10, Definition 1.1], [DeT17, Section 2.3], and [DoT23, Chapter 1.4]).

Let (X, ρ) be a compact metric space, $T\colon X \to X$ be a Borel-measurable transformation such that $\mathcal{M}(X, T) \neq \emptyset$, and $\phi\colon X \to [-\infty, +\infty]$ be a Borel function. Then the *topological pressure* of the potential ϕ with respect to the transformation T is given by

$$P(T, \phi) := \sup\bigl\{h_\mu(T) + \langle \mu, \phi\rangle : \mu \in \mathcal{M}(X, T) \text{ and } \langle \mu, \phi\rangle > -\infty\bigr\}. \tag{4.2}$$

A measure $\mu \in \mathcal{M}(X, T)$ that attains the supremum in (4.2) is called an *equilibrium state* for the transformation T and the potential ϕ. Denote the set of all such measures by $\mathcal{E}(T, \phi)$. In particular, when the potential ϕ is the constant function 0, we denote $h_{\mathrm{top}}(T) := P(T, 0)$ and say that a measure $\mu \in \mathcal{M}(X, T)$ is a *measure of maximal entropy* of T if $\mu \in \mathcal{E}(T, 0)$.

Definition 4.1 (Jacobian). Let (X, ρ) be a compact metric space, and $T\colon X \to X$ be a Borel-measurable transformation. We say that $A \subseteq X$ is *admissible* (for T) if A, $T(A) \in \mathcal{B}(X)$, and

$T|_A$ is injective. Suppose $J\colon X \to [0, +\infty)$ is a Borel function, $\mu \in \mathcal{P}(X)$, and $E \in \mathcal{B}(X)$ with $\mu(E) = 1$. Then J is said to be a *Jacobian on* E for T with respect to μ if for all admissible sets $A \subseteq E$,

$$\mu(T(A)) = \int_A J \, \mathrm{d}\mu.$$

Moreover, we say that J is a *Jacobian* for T with respect to μ if there exists $\widetilde{E} \in \mathcal{B}(X)$ with $\mu\bigl(\widetilde{E}\bigr) = 1$ such that J is a Jacobian on $\widetilde{E}$ for T with respect to μ.

Recall that $\mathcal{P}(X;Y) = \{\mu \in \mathcal{P}(X) : \mu(Y) = 1\}$ for $Y \in \mathcal{B}(X)$. We state below the hypotheses under which we will develop our theory in this section.

Definition 4.2. We say that the sextuple $(X,\, \rho,\, T,\, Y,\, \{Y_k\}_{k\in\mathbb{N}},\, \mu)$ is *admissible* if it has the following properties:

(i) (X, ρ) is a compact metric space.
(ii) $T\colon X \to X$ is a Borel-measurable transformation.
(iii) $\{Y_k\}_{k\in\mathbb{N}}$ is a sequence of pairwise disjoint admissible sets for T.
(iv) $Y = \bigcup_{k\in\mathbb{N}} Y_k$.
(v) $\mu \in \mathcal{M}(X,T) \cap \mathcal{P}(X;Y)$.

The following proposition states the uniqueness of the Jacobian and provides a lower bound for the measure-theoretic entropy in terms of the Jacobian.

Proposition 4.3. *Let $(X,\, \rho,\, T,\, Y,\, \{Y_k\}_{k\in\mathbb{N}},\, \mu)$ be admissible. Assume that $J\colon X \to [0, +\infty)$ is a Jacobian for T with respect to μ. Then $J(x) \geqslant 1$ for μ-a.e. $x \in X$, and $h_\mu(T) \geqslant \langle \mu, \log(J) \rangle$. Moreover, for each Borel function $\widetilde{J}\colon X \to [0, +\infty)$, $\widetilde{J}$ is a Jacobian for T with respect to μ if and only if $J(x) = \widetilde{J}(x)$ for μ-a.e. $x \in X$.*

The lower bound given above is a classical result in ergodic theory known as the Rokhlin entropy formula. We refer the reader to [Sa99, Theorem 4.2] for a version for topological Markov shifts and to [Co12, Corollary 12.1] for a version for finite admissible partitions. The uniqueness of the Jacobian immediately follows from [Ro49, Theorem 2.7], [PU10, Definition 2.9.2 & Proposition 2.9.5].

Proposition 4.3 is the so-called Rokhlin entropy formula (see [Sa99, Theorem 4.2] for its topological Markov shift version and [Sa99, Section 4.1.3] for its proof). Now we establish it in our context.

Let $\bigl(X,\, \rho,\, T,\, Y,\, \{Y_k\}_{k\in\mathbb{N}},\, \mu\bigr)$ be admissible. Let $\overline{\mu}$ be the completion of μ. Then by [Ro49, Theorem 2.7], the compact metric space (X, ρ) equipped the complete measure $\overline{\mu}$ is a Lebesgue space in the sense of V. Rokhlin. Moreover, according to [PU10, Definition 2.9.2] (or an equivalent definition of *essentially countable to one* endomorphisms in [Pa69, Subsection 10.1]), T is essentially countable to one. We omit V. Rokhlin's definition of Lebesgue spaces and essentially countable to one endomorphisms here and refer the reader to [Ro49, Section 2], since the only result we will use about them are the following result on Jacobians.

Proposition 4.4. *Let $(X,\, \mathcal{B},\, \mu)$ be a Lebesgue space, $T : X \to X$ be an essentially countable to one endomorphism, and $\mu \in \mathcal{M}(X,T)$. Let $J\colon X \to [0, \infty)$ be a Jacobian for T with respect to μ. Then $H_\mu(\epsilon | T^{-1}(\epsilon)) = \int \log J \, \mathrm{d}\mu$, where ϵ denotes the point partition of X, and $H_\mu(\epsilon | T^{-1}(\epsilon))$ denotes the conditional entropy of ϵ given the smallest σ-algebra which contains $T^{-1}(\epsilon)$.*

This follows from [PU10, Theorem 2.9.6], [Pa69, Lemma 10.5], and [Pa69, Definition 4.2]. Now we complete the proof of Proposition 4.3.

Proof of Proposition 4.3. First, we show that $J(x) \geqslant 1$ for μ-a.e. $x \in X$. Indeed, Since J is a Jacobian for T with respect to μ, by Definition 4.1, there exists $E \in \mathcal{B}(X)$ with $\mu(E) = 1$ such that J is a Jacobian on E for T with respect to μ. Hence, since $\mu \in \mathcal{M}(X,T)$ and $Y = \bigcup_{k\in\mathbb{N}} Y_k$, for each Borel $A \subseteq E \cap Y$, we have $\int_A J \,\mathrm{d}\mu = \sum_{k\in\mathbb{N}} \int_{A\cap Y_k} J \,\mathrm{d}\mu = \sum_{k\in\mathbb{N}} \mu(T(A\cap Y_k)) \geqslant \sum_{k\in\mathbb{N}} \mu(A\cap Y_k) = \mu(A)$. Hence, $J(x) \geqslant 1$ for μ-a.e. $x \in E \cap Y$. Then by $\mu \in \mathcal{P}(X;Y)$, we obtain that $J(x) \geqslant 1$ for μ-a.e. $x \in X$.

Now we show $h_\mu(T) \geqslant \langle \mu, \log(J)\rangle$. Let $\overline{\mu}$ be the completion of the measure μ. Then by Definition 4.1, $\overline{\mu} \in \mathcal{M}(X,T)$ and J is a Jacobian for T with respect to $\overline{\mu}$. Since (X, ρ) is compact metric space, by [Ro49, Theorem 2.7], $(X, \mathcal{B}, \overline{\mu})$ is a Lebesgue space, where $\mathcal{B}$ denotes the σ-algebra containing all $\overline{\mu}$-measurable sets. Since T satisfies Assumption 1 (ii) and (iii), by [PU10, Definition 2.9.2], T is an essentially countable to one endomorphism. By Proposition 4.4, we have $H_{\overline{\mu}}(\epsilon|T^{-1}(\epsilon)) = \langle \overline{\mu}, \log(J)\rangle = \langle \mu, \log(J)\rangle$. Note that ϵ is an invariant partition, i.e., $T^{-1}(\epsilon)$ is coarser than ϵ. Then by [Ro67, Section 7], we have $h_{\overline{\mu}}(T,\epsilon) = H_{\overline{\mu}}(\epsilon|T^{-1}(\epsilon))$. By [Ro67, Section 9], $h_{\overline{\mu}}(T) \geqslant h_{\overline{\mu}}(T,\epsilon)$. Since $\overline{\mu}$ is the completion of the measure μ, by (4.1), we have $h_\mu(T) = h_{\overline{\mu}}(T)$. Thus, we obtain that $h_\mu(T) = h_{\overline{\mu}}(T) \geqslant h_{\overline{\mu}}(T,\epsilon) = H_{\overline{\mu}}(\epsilon|T^{-1}(\epsilon)) = \langle \mu, \log(J)\rangle$.

Next, we assume that $\widetilde{J}\colon X \to [0,+\infty)$ is a Jacobian on $\widetilde{E}$ for T with respect to μ. Then we have $\int_A J \,\mathrm{d}\mu = \mu(T(A)) = \int_A \widetilde{J} \,\mathrm{d}\mu$ for each admissible $A \subseteq E \cap \widetilde{E}$. Hence, by $\mu(E \cap \widetilde{E}) = 1$, we obtain that $J(x) = \widetilde{J}(x)$ for μ-a.e. $x \in X$.

Finally, we assume that $J(x) = \widetilde{J}(x)$ for μ-a.e. $x \in X$. Then for each admissible $A \subseteq E$, we have $\mu(T(A)) = \int_A J \,\mathrm{d}\mu = \int_A \widetilde{J} \,\mathrm{d}\mu$. Thus, $\widetilde{J}$ is a Jacobian for T with respect to μ. $\square$

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