COMPUTABILITY OF EQUILIBRIUM STATES IN HYPERBOLIC SYSTEMS AND BEYOND.

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ABSTRACT. We invesigate the computability of the equilibrium states for a class of nonuniformly expanding local diffeomorphisms on smooth manifolds and Hölder continuous potentials with not very large oscillations.

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1. INTRODUCTION

1.1. **Main results.** We study a class of non-uniformly expanding maps considered by Castro and Varandas in [CV13]. A list of assumptions that are applied in the rest of this paper is displayed below:

The Assumptions.

(i) M is a compact and connected Riemann manifold with distance d.

(ii) The map $f\colon M \to M$ is a local homeomorphism of M with degree $\deg(f)$.

(iii) Two continuous functions $L\colon M \to \mathbb{R}^+$ and $r\colon M \to \mathbb{R}^+$ satisfy that, for each $x \in M$, $f_x \coloneqq f|_{B(x,r(x))}\colon B(x, r(x)) \to f(B(x, r(x)))$, the restriction of f to $B(x, r(x))$, is invertible and satisfies that
$$d(f_x^{-1}(y), f_x^{-1}(z)) \leqslant L(x)\, d(y, z) \quad \text{for each pair of } y,\, z \in f(B(x, r(x))).$$

(iv) There exists a pair of constants $\sigma > 1$ and $L \in (1, 2)$, and an open subset $\mathcal{A} \subseteq M$ satisfying that $L(x) \leqslant L$ for each $x \in \mathcal{A}$ and $L(x) < \sigma^{-1}$ for each $x \notin \mathcal{A}$.

(v) There exists a finite covering $\{U_i\}_{i=1}^n$ of M by open subsets of injectivity for f such that there exists an integer $q < \deg(f)$ with $\mathcal{A} \subseteq \bigcup_{i=1}^q U_i$.

Under the Assumptions, for each $\delta > 0$, we say that δ is *acceptable* if for each pair $x, y \in M$ with $d(x, y) < \delta$ and an enumeration $\{y_i : 1 \leqslant i \leqslant \deg(f)\}$ of $f^{-1}(\{y\})$, there exists a corresponding enumeration $\{x_i : 1 \leqslant i \leqslant \deg(f)\}$ of $f^{-1}(\{x\})$ such that for these pairs $\{(x_i, y_i)\}_{i=1}^{\deg(f)}$ of paired preimages associated to x and y, the following statements are true:

(i) For each integer $1 \leqslant i \leqslant \deg(f)$, $d(x_i, y_i) \leqslant Ld(x, y)$.

(ii) There are at least $(\deg(f)-q)$ pairs (x_k, y_k) satisfy a sharper inequality: $d(x_k, y_k) \leqslant \sigma^{-1}d(x, y)$.

According to [CV13, Lemma 3.10], we obtain the existence of acceptable constant δ. Now we state the following hypothese on the basis of the Assumptions.

The Additional Assumptions.

(vi) There exist two constants $c > 0$ and $0 < \gamma < 1$ with $\sigma^{-(1-\gamma)}L^{\gamma} < e^{-2c}$.

(vii) The constant $\delta > 0$ is acceptable.

(viii) There exists a sequence $\{x_i\}_{i=1}^{m-1}$ of points in M such that $\bigcup_{i=1}^{m-1} B(x_i, \delta/3) = M$.

(ix) The constant $\varepsilon_\phi \in (0, \log(\deg(f)) - \log(q))$ satisfies that

$$e^{\varepsilon_\phi}\left(\frac{(\deg(f) - q)\sigma^{-\alpha} + qL^{\alpha}(1 + (L - 1)^{\alpha})}{\deg(f)}\right) + 2mL^{\alpha}\varepsilon_\phi \cdot \operatorname{diam}(M)^{\alpha} < 1.$$

(x) The function $\phi \in C^{0,\alpha}(M, d)$ satisfies that

$$\sup_{z\in M} \phi(z) - \inf_{z\in M} \phi(z) < \varepsilon_\phi \quad \text{and} \quad |\exp(\phi)|_\alpha < \varepsilon_\phi \exp\big(\inf_{z\in M} \phi(z)\big).$$

We recall [CV13, Theorem A] here.

Theorem 1.1. *Under the Assumptions and the Additional Assumptions, there exists a unique equilibrium state $\mu_{f,\phi}$ for f and ϕ.*

Now we concern the computability of equilibrium state $\mu_{f,\phi}$ for f and ϕ.

Theorem 1.2. *Let $(M, d, \mathcal{S})$ be a computable metric space in which M is recursively compact. Then there exists an algorithm that satisfies the following property:*

For each M, $f(x)$, $L(x)$, $r(x)$, σ, L, $\{U_i\}_{i=1}^n$, q, ε_ϕ, $\phi(x)$ satisfying the Assumptions and the Additional Assumptions and each $t \in \mathbb{N}$, this algorithm outputs a rational linear combination of finite dirac measures which are supported on some points in $\mathcal{S}$ as a 2^{-t}-approximation in the Wasserstein–Kantorovich metric W_d for the unique equilibrium state $\mu_{f,\phi}$ for f and ϕ, after inputting the following data in this algorithm:

(i) *an algorithm computing the function $\phi\colon M \to \mathbb{R}$,*

(ii) *an algorithm computing the map $f\colon M \to M$,*

(iii) *an algorithm computing the function $r\colon M \to \mathbb{R}^+$,*

(iv) *an algorithm computing σ, L, and ε_ϕ,*

(v) *the integers t, $\deg(f)$, and q.*

2. Notation

We use $\mathbb{N}$ to denote the set of integers greater than or equal to 1 and $\mathbb{N}^* := \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$. We write $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $\mathbb{N}_0^* := \{0\} \cup \mathbb{N}^*$. We denote by $\mathbb{Q}^+$ (resp. $\mathbb{R}^+$) the set of all positive rational (resp. real) numbers. The symbol log denotes the natural logarithm. For $x \in \mathbb{R}$, we define $\lfloor x \rfloor$ as the greatest integer $\leqslant x$, and $\lceil x \rceil$ as the smallest integer $\geqslant x$. The cardinality of a set A is denoted by $\operatorname{card} A$.

Consider a map $f\colon X \to X$ on a set X. We write f^n for the n-th iterate of f, and $f^{-n} := (f^n)^{-1}$, for each $n \in \mathbb{N}$. We set $f^0 := \mathrm{id}_X$, the identity map on X. For a real-valued function $\phi\colon X \to \mathbb{R}$, we write $S_n\phi(x) = S_n^f\phi(x) := \sum_{m=0}^{n-1} \phi(f^m(x))$ for $x \in X$ and $n \in \mathbb{N}_0$. We omit the superscript f when the map f is clear from the context. When $n = 0$, by definition $S_0\phi = 0$.

Let (X, d) be a metric space. We denote by $\mathcal{B}(X)$ the σ-algebra of all Borel subsets of X. For each subset $Y \subseteq X$, we denote the diameter of Y by $\operatorname{diam}_d Y := \sup\{d(x, y) : x, y \in Y\}$, the interior of Y by $\operatorname{int}_\circ Y$, and the characteristic function of Y by $\mathbb{1}_Y$.

For each $r \in \mathbb{R}$ and each $x \in X$, we denote the open (resp. closed) ball of radius r centered at x by $B_d(x, r) := \{y \in X : d(x, y) < r\}$. For each $r \in \mathbb{R}$ and each nonempty set $K \subseteq X$, we define

$d(x,K) \coloneqq \inf_{y\in K} d(x,y)$, and $B_d(K,r) \coloneqq \{x \in X : d(x,K) < r\}$. We often omit the metric d in the subscript when it is clear from the context.

For a compact metric space (X,d), we denote by $C(X)$ the space of continuous functions from X to $\mathbb{R}$, and by $\mathcal{M}(X)$ (resp. $\mathcal{P}(X)$) the set of finite signed Borel measures (resp. Borel probability measures) on X. Let $g\colon X \to X$ be a Borel-measurable transformation. We denote by $\mathcal{M}(X,g)$ the set of g-invariant Borel probability measures on X. Moreover, for each Borel subset $C \in \mathcal{B}(X)$, $\mathcal{P}(X;C)$ denotes the set $\{\mu \in \mathcal{P}(X) : \mu(C) = 1\}$. By the Riesz representation theorem, we can identify the dual of $C(X)$ with the space $\mathcal{M}(X)$. For $\mu \in \mathcal{M}(X)$, we use $\|\mu\|$ to denote the total variation norm of μ, $\operatorname{supp}\mu$ to denote the support of μ, and

$$\langle \mu, u\rangle \coloneqq \int u \, \mathrm{d}\mu$$

for each μ-integrable Borel function u on X. If we do not specify otherwise, we equip $C(X)$ with the uniform norm $\|\cdot\|_{C(X)} \coloneqq \|\cdot\|_\infty$, and equip $\mathcal{M}(X)$, $\mathcal{P}(X)$, and $\mathcal{M}(X,g)$ with the weak* topology.

The space of real-valued Hölder continuous functions with an exponent $\alpha \in (0,1]$ on a compact metric space (X,d) is denoted as $C^{0,\alpha}(X,d)$. For each $\phi \in C^{0,\alpha}(X,d)$,

$$|\phi|_\alpha \coloneqq \sup\{|\phi(x) - \phi(y)|/d(x,y)^\alpha : x,\, y \in X,\, x \neq y\}. \tag{2.1}$$

For a complete separable metric space (X,d), we recall the Wasserstein–Kantorovich metric W_d on $\mathcal{P}(X)$ given by

$$W_d(\mu,\nu) \coloneqq \sup\bigl\{|\langle \mu, f\rangle - \langle \nu, f\rangle| : f \in C^{0,1}(X,d),\, |f|_1 \leqslant 1\bigr\}. \tag{2.2}$$

Note that for Borel probability measures in $\mathcal{P}(X)$, the convergence in W_d is equivalent to the convergence in the weak* topology (see e.g. [Vi09, Corollary 6.13]).

3. Preliminaries

3.1. **Computable analysis.** We recall fundamental notions and results from recursion theory and computable analysis.[1] We present, in order, definitions and results concerning the computability of real numbers, computable structures on metric spaces, computability of open sets, functions, compact sets, and probability measures.

Computability over the reals. We begin by reviewing basic notations and concepts from classical recursion theory; for an introduction, see e.g. [Bri94, Chapter 3].

Definition 3.1 (Effective enumeration and recursively enumerable set). Let $S \subseteq \mathbb{N}^*$ be a nonempty set. An *effective enumeration* of S is a sequence $\{x_i\}_{i\in\mathbb{N}}$ with $S = \{x_i : i \in \mathbb{N}\}$ such that there exists an algorithm that, for each $i \in \mathbb{N}$, upon input i, outputs x_i.

Moreover, a set $I \subseteq \mathbb{N}^*$ is said to be a *recursively enumerable set*[2] if $I = \emptyset$ or there exists an effective enumeration of I.

For brevity, the symbol I denotes a nonempty recursively enumerable set throughout this subsection. Note that $\mathbb{N}^k$, for $k \in \mathbb{N}$, and $\mathbb{N}^*$ are all recursively enumerable sets by Definition 3.1.

Definition 3.2 (Partial recursive and recursive function). Let $\{i_n\}_{n\in\mathbb{N}}$ be an effective enumeration of I. We say that $f\colon I \to \mathbb{N}_0^*$ is *partial recursive* if there exists an algorithm that, for each $n \in \mathbb{N}$, on input n, outputs $f(i_n)$ if $f(i_n) \in \mathbb{N}^*$, and runs forever otherwise, namely, if $f(i_n) = 0$. We say that $f\colon I \to \mathbb{N}_0^*$ is *recursive* if f is a partial recursive function with $f(I) \subseteq \mathbb{N}^*$.

We now define the computability of real numbers.

[1]Our notion of algorithm is consistent with *Type-2 machines* defined in [We00, Definition 2.1.1].

[2]We emphasize that recursively enumerable sets in this article are subsets of $\mathbb{N}^*$.

Definition 3.3 (Computable real number). A real number x is called *computable* if there exist three recursive functions $f\colon \mathbb{N} \to \mathbb{N}$, $g\colon \mathbb{N} \to \mathbb{N}$, and $h\colon \mathbb{N} \to \mathbb{N}$ such that $\left|(-1)^{h(n)} f(n)/g(n) - x\right| < 2^{-n}$ for all $i \in I$ and $n \in \mathbb{N}$.

Let $\{x_i\}_{i\in I}$ be a sequence of real numbers. We say that $\{x_i\}_{i\in I}$ is a *sequence of uniformly computable real numbers* if there exist three recursive functions $f\colon \mathbb{N} \times I \to \mathbb{N}$, $g\colon \mathbb{N} \times I \to \mathbb{N}$, and $h\colon \mathbb{N} \times I \to \mathbb{N}$ such that $\left|(-1)^{h(n,i)} f(n,i)/g(n,i) - x_i\right| < 2^{-n}$ for all $i \in I$ and $n \in \mathbb{N}$.

Clearly, $x \in \mathbb{R}$ is computable if and only if $\{x_i\}_{i\in\mathbb{N}}$ defined by $x_i \coloneqq x$ for all $i \in \mathbb{N}$ is uniformly computable. For analogous concepts in the sequel, we will define the uniform sequence version and regard the individual case as the special case of constant sequences.

Computable metric spaces.

Definition 3.4 (Computable metric space). A *computable metric space* is a triple $(X, \rho, \mathcal{S})$ satisfying that

(i) (X, ρ) is a separable metric space,
(ii) $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$ forms a countable dense subset $\{s_n : n \in \mathbb{N}\}$ of X, and
(iii) $\{\rho(s_m, s_n)\}_{(m,n)\in\mathbb{N}^2}$ is a sequence of uniformly computable real numbers.

The points in $\mathcal{S}$ are called *ideal*. Since $\mathbb{N}^3$ is recursively enumerable, the collection $\mathcal{B} \coloneqq \{B(s_i, m/n) : i, m, n \in \mathbb{N}\}$ can be enumerated as $\{B_l\}_{l\in\mathbb{N}}$ satisfying the following: there exists an algorithm that, for each $l \in \mathbb{N}$, upon input l, outputs $i, m, n \in \mathbb{N}$ with $B_l = B(s_i, m/n)$. We call the elements in $\mathcal{B}$ *ideal balls* and such an enumeration of $\mathcal{B}$ an *effective enumeration of ideal balls* in $(X, \rho, \mathcal{S})$.

We then define the computability of points in a computable metric space.

Definition 3.5 (Computable point). Let $(X, \rho, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$, and $\{x_i\}_{i\in I}$ be a sequence of points in X. Then $\{x_i\}_{i\in I}$ is called *uniformly computable* (in $(X, \rho, \mathcal{S})$) if there exists a recursive function $f\colon \mathbb{N} \times I \to \mathbb{N}$ such that $\rho\big(s_{f(n,i)}, x_i\big) < 2^{-n}$ for all $n \in \mathbb{N}$ and $i \in I$. Moreover, a point x in X is *computable* (in $(X, \rho, \mathcal{S})$) if $\{x_i\}_{i\in\mathbb{N}}$ defined by $x_i \coloneqq x$ for all $i \in \mathbb{N}$ is uniformly computable.

We now specify the computable structure on $\mathbb{R}$. Let $\mathcal{S}_{\mathbb{Q}} = \{q_n\}_{n\in\mathbb{N}}$ be the enumeration of $\mathbb{Q}$ induced by an effective enumeration of $\mathbb{N}^3$ via the mapping $(a, b, c) \mapsto (-1)^c a/b$. Note that $\{d_{\mathbb{R}}(q_m, q_n)\}_{(m,n)\in\mathbb{N}^2}$ is a sequence of uniformly computable real numbers, where $d_{\mathbb{R}}$ is the Euclidean metric. Then the triple $\big(\mathbb{R}, d_{\mathbb{R}}, \mathcal{S}_{\mathbb{Q}}\big)$ forms a computable metric space according to Definition 3.4. A similar construction provides a computable structure for $\mathbb{R}^+$. In this article, we fix these as the standard computability structures on $\mathbb{R}$ and $\mathbb{R}^+$. It is clear that under these structures, Definitions 3.3 and 3.5 are equivalent for the computability of real numbers. That is, a sequence of reals is uniformly computable in one sense if and only if it is in the other.

We also consider a weaker notion of computability over $\mathbb{R}$ that leverages its natural ordered structure.

Definition 3.6 (Semi-computable real number). Let $\{x_i\}_{i\in I}$ be a sequence of real numbers. We say that $\{x_i\}_{i\in I}$ is *uniformly lower* (resp. *upper*) *semi-computable* if there exist three recursive functions $f\colon \mathbb{N} \times I \to \mathbb{N}$, $g\colon \mathbb{N} \times I \to \mathbb{N}$, and $h\colon \mathbb{N} \times I \to \mathbb{N}$ such that for each $i \in I$, $\big\{(-1)^{h(n,i)} f(n,i)/g(n,i)\big\}_{n\in\mathbb{N}}$ is non-decreasing (resp. non-increasing) and converges to x_i as $n \to +\infty$. Moreover, a real number x is called *lower* (resp. *upper*) *semi-computable* if the sequence $\{x_i\}_{i\in\mathbb{N}}$ defined by $x_i \coloneqq x$ for each $i \in \mathbb{N}$ is uniformly lower (resp. upper) semi-computable.

Lower semi-computable open sets. We define an effective version of open sets and collect some relevant results.

Let $(X, \rho, \mathcal{S})$ be a computable metric space. Let $\mathcal{B}$ be the set of ideal balls, and $\{B_l\}_{l\in\mathbb{N}}$ be an effective enumeration of ideal balls in $(X, \rho, \mathcal{S})$. We define the set $\mathcal{B}_0 \coloneqq \mathcal{B} \cup \{\emptyset\}$ of *extended ideal balls* and an enumeration $\{D_l\}_{l\in\mathbb{N}}$ of $\mathcal{B}_0$ such that $D_1 = \emptyset$ and $D_l = B_{l-1}$ for each integer $l \geqslant 2$. We call such an enumeration an *effective enumeration of extended ideal balls* in $(X, \rho, \mathcal{S})$.

Definition 3.7 (Lower semi-computable open set). Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space, and $\{D_l\}_{l\in\mathbb{N}}$ be an effective enumeration of extended ideal balls. Then a sequence $\{U_i\}_{i\in I}$ of open sets in X is said to be *uniformly lower semi-computable open* (in $(X,\,\rho,\,\mathcal{S})$) if there exists a recursive function $f\colon \mathbb{N}\times I\to\mathbb{N}$ such that $U_i=\bigcup_{n\in\mathbb{N}} D_{f(n,i)}$ for each $i\in I$. Moreover, an open set $U\subseteq X$ is called *lower semi-computable open* (in $(X,\,\rho,\,\mathcal{S})$) if the sequence $\{U_i\}_{i\in\mathbb{N}}$ defined by $U_i \coloneqq U$ for $i\in\mathbb{N}$ is uniformly lower semi-computable open.

The above definition of a lower semi-computable open set differs slightly from the ones in [BBRY11, Definition 3.4] and [BRY14, Definition 2.4]. In our definition, we use extended ideal balls, which include the empty set $\emptyset$.

The term *recursively open set* in the literature (e.g. [GHR11, Subsection 2.2 and Definition 2.4] and [HR09, Subsection 3.3]) is equivalent to the notion of *lower semi-computable open set* defined above. A detailed discussion of this equivalence is provided in [He25, Subsection 3.3].

Note that we can algorithmically decide whether $s\in B$ for each ideal point $s\in\mathcal{S}$ and each extended ideal ball $B\in\mathcal{B}_0$. The following result then follows immediately from Definition 3.7 (see e.g. [He25, Proposition 3.9]).

Proposition 3.8. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space with $\mathcal{S}=\{s_n\}_{n\in\mathbb{N}}$. Assume that $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open. Then there exists a recursively enumerable set $E\subseteq\mathbb{N}\times I$ such that $\{s_n : (n,i)\in E_i\}=\{s_n : n\in\mathbb{N}\}\cap U_i$, where $E_i\coloneqq\{(n,i)\in E : n\in\mathbb{N}\}$ for each $i\in I$.*

The following two results are two classical results in computable analysis which both follow immediately from Definitions 3.1 and 3.7 (see e.g. [He25, Propositions 3.10 & 3.11]).

Proposition 3.9. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. Assume that H and L are two nonempty recursively enumerable sets with $L\subseteq I\times H$, and that $\{U_{i,h}\}_{(i,h)\in L}$ is uniformly lower semi-computable open. Then $\{\bigcup\{U_{i,h} : (i,h)\in L_h\}\}_{h\in H}$ is uniformly lower semi-computable open, where $L_h\coloneqq\{(i,h)\in L : i\in I\}$ for each $h\in H$. In particular, if $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open, then $\bigcup_{i\in I} U_i$ is lower semi-computable open.*

Proposition 3.10. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. Assume that $\{r_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable real numbers and $\{x_i\}_{i\in I}$ is uniformly computable in $(X,\,\rho,\,\mathcal{S})$. Then $\{B(x_i,r_i)\}_{i\in I}$ is uniformly lower semi-computable open.*

Computability of functions. We begin with the definition of oracles for points.

Definition 3.11 (Oracle). Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space with $\mathcal{S}=\{s_i\}_{i\in\mathbb{N}}$, and $x\in X$. We say that a function $\tau\colon\mathbb{N}\to\mathbb{N}$ is an *oracle* for x if $\rho(s_{\tau(n)},x)<2^{-n}$ for each $n\in\mathbb{N}$.

With the above definition, computable functions can be defined as follows.

Definition 3.12 (Computable function). Let $(X,\,\rho,\,\mathcal{S})$ and $(X',\,\rho',\,\mathcal{S}')$ be computable metric spaces with $\mathcal{S}=\{s_n\}_{n\in\mathbb{N}}$ and $\mathcal{S}'=\{s'_n\}_{n\in\mathbb{N}}$. Assume that $\{i_n\}_{n\in\mathbb{N}}$ is an effective enumeration of I, and $C_i\subseteq X$ for each $i\in I$. Then a sequence $\{f_i\}_{i\in I}$ of functions $f_i\colon X\to X'$ is called a *sequence of uniformly computable functions with respect to* $\{C_i\}_{i\in I}$ if there exists an algorithm that, for all $l,\,n\in\mathbb{N}$, $x\in C_{i_n}$, and oracle τ for x, on input l, n, and τ, outputs $m\in\mathbb{N}$ with $\rho'(s'_m,f_{i_n}(x))<2^{-l}$. We often omit the phrase "with respect to $\{C_i\}_{i\in I}$" when $C_i=X$ for all $i\in I$. Moreover, a function $f\colon X\to X'$ is said to be a *computable function on* C if $\{f_i\}_{i\in\mathbb{N}}$, defined by $f_i\coloneqq f$ for all $i\in\mathbb{N}$, is a sequence of uniformly computable functions with respect to $\{C_i\}_{i\in\mathbb{N}}$ defined by $C_i\coloneqq C$ for all $i\in\mathbb{N}$. We often omit the phrase "with respect to C" when $C=X$.

Computable functions serve as an effective version of continuous functions. The following result provides examples of computable functions (see e.g. [We00, Examples 4.3.3 and 4.3.13.5]).

Example 3.13. The exponential function $\exp\colon\mathbb{R}\to\mathbb{R}$ and the logarithmic function $\log\colon\mathbb{R}^+\to\mathbb{R}$ are computable functions.

We recall the following classical characterization of computable functions (cf. [RY21a, Proposition 5.2.14] and [BBRY11, Proposition 3.6]; see also [He25, Proposition 3.17]).

Proposition 3.14. *Let $(X,\, \rho,\, \mathcal{S})$ and $(X',\, \rho',\, \mathcal{S}')$ be computable metric spaces. Suppose $\{B'_n\}_{n\in\mathbb{N}}$ is an effective enumeration of ideal balls in $(X',\, \rho',\, \mathcal{S}')$. Given $f_i \colon X \to X'$ and $C_i \subseteq X$ for each $i \in I$, the following statements are equivalent:*

(i) *The sequence $\{f_i\}_{i\in I}$ is a sequence of uniformly computable functions with respect to $\{C_i\}_{i\in I}$.*

(ii) *There exists a sequence $\{U_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ of uniformly lower semi-computable open sets in $(X,\, \rho,\, \mathcal{S})$ such that $f_i^{-1}(B'_n) \cap C_i = U_{n,i} \cap C_i$ for all $i \in I$ and $n \in \mathbb{N}$.*

(iii) *For each nonempty recursively enumerable set M and each sequence $\{V'_m\}_{m\in M}$ of uniformly lower semi-computable open sets, there exists a sequence $\{W_{m,i}\}_{(m,i)\in M\times I}$ of uniformly lower semi-computable open sets in $(X,\, \rho,\, \mathcal{S})$ such that $f_i^{-1}(V'_m) \cap C_i = W_{m,i} \cap C_i$ for all $m \in M$ and $i \in I$.*

We now define a notion of weaker computability property for functions.

Definition 3.15 (Semi-computable function). Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, $\{i_n\}_{n\in\mathbb{N}}$ be an effective enumeration of I, and $C_i \subseteq X$ for each $i \in I$. A sequence $\{f_i\}_{i\in I}$ of functions $f_i \colon X \to \mathbb{R}$ is a *sequence of uniformly upper* (resp. *lower*) *semi-computable functions with respect to* $\{C_i\}_{i\in I}$ if there exists an algorithm that, for all $l,\, n \in \mathbb{N}$, $x \in C_{i_n}$, and oracle τ for x, on input l, n, and τ, outputs $q_{l,n,\tau} \in \mathbb{Q}$ such that for each $n \in \mathbb{N}$, each $x \in C_{i_n}$, and each oracle τ for x, $\{q_{l,n,\tau}\}_{l\in\mathbb{N}}$ is non-increasing (resp. non-decreasing) and converges to $f_{i_n}(x)$ as $l \to +\infty$. We often omit the phrase "with respect to $\{C_i\}_{i\in I}$" when $C_i = X$ for each $i \in I$. Moreover, a function $f \colon X \to \mathbb{R}$ is said to be an *upper* (resp. a *lower*) *semi-computable function on C* if $\{f_i\}_{i\in\mathbb{N}}$ defined by $f_i \coloneqq f$ for each $i \in \mathbb{N}$, is a sequence of uniformly upper (resp. lower) semi-computable functions with respect to $\{C_i\}_{i\in\mathbb{N}}$ defined by $C_i \coloneqq C$ for all $i \in \mathbb{N}$. We often omit the phrase "with respect to C" when $C = X$.

The following proposition is an immediate consequence of Proposition 3.14 (see e.g. [He25, Proposition 3.19]).

Proposition 3.16. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, and $\mathcal{S}_{\mathbb{Q}} = \{q_n\}_{n\in\mathbb{N}}$. Given $f_i \colon X \to \mathbb{R}$ and $C_i \subseteq X$ for all $i \in I$, the following statements are equivalent:*

(i) *The sequence $\{f_i\}_{i\in I}$ is a sequence of uniformly upper (resp. lower) semi-computable functions with respect to $\{C_i\}_{i\in I}$.*

(ii) *There exists a sequence $\{U_{n,i}\}_{(n,i)\in\mathbb{N}\times I}$ of uniformly lower semi-computable open sets in $(X,\, \rho,\, \mathcal{S})$ such that $f_i^{-1}(Q_n) \cap C_i = U_{n,i} \cap C_i$ with $Q_n \coloneqq (-\infty, q_n)$ (resp. $Q_n \coloneqq (q_n, +\infty)$) for all $i \in I$ and $n \in \mathbb{N}$.*

(iii) *For each nonempty recursively enumerable set L and each sequence $\{r_l\}_{l\in L}$ of uniformly computable real numbers, there exists a sequence $\{W_{l,i}\}_{(l,i)\in L\times I}$ of uniformly lower semi-computable open sets in $(X,\, \rho,\, \mathcal{S})$ such that $f_i^{-1}(R_l) \cap C_i = W_{l,i} \cap C_i$ with $R_l \coloneqq (-\infty, r_l)$ (resp. $R_l \coloneqq (r_l, +\infty)$) for all $l \in L$ and $i \in I$.*

Recursively compact sets and recursively precompact metric spaces. Here we recall the definitions of recursive compactness and recursive precompactness. For a more detailed discussion, see [GHR11, Section 2].

Definition 3.17 (Recursively compact set). Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$, and $\{i_l\}_{l\in\mathbb{N}}$ be an effective enumeration of I. A sequence $\{K_i\}_{i\in I}$ of compact sets in X is called *uniformly recursively compact* (in $(X,\, \rho,\, \mathcal{S})$) if there exists an algorithm that, for each $n \in \mathbb{N}$, each sequence $\{m_n\}_{n=1}^{p}$ of integers, and each sequence $\{q_n\}_{n=1}^{p}$ of positive rational numbers, upon input, halts if and only if $K_{i_l} \subseteq \bigcup_{n=1}^{p} B(s_{m_n}, q_n)$. Moreover, a set $K \subseteq X$ is called *recursively compact* (in $(X,\,\rho,\,\mathcal{S})$) if the sequence $\{K_i\}_{i\in\mathbb{N}}$ defined by $K_i \coloneqq K$ for each $i \in \mathbb{N}$, is uniformly recursively compact.

Note that for each compact set K and each function $f\colon \mathbb{N} \to \mathbb{N}$, $K \subseteq \bigcup_{n\in\mathbb{N}} D_{f(n)}$ if and only if $K \subseteq \bigcup_{n=1}^{k} D_{f(n)}$ for some $k \in \mathbb{N}$. This implies the following result.

Proposition 3.18. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. Suppose $\{h_m\}_{m\in\mathbb{N}}$ (resp. $\{l_n\}_{n\in\mathbb{N}}$) is an effective enumeration of a nonempty recursively enumerable set H (resp. L). Assume that $\{K_h\}_{h\in H}$ is uniformly recursively compact and $\{U_l\}_{l\in L}$ is uniformly lower semi-computable open. Then there exists an algorithm that, for all $m,\,n \in \mathbb{N}$, upon input, halts if and only if $K_{h_m} \subseteq U_{l_n}$.*

We collect some fundamental properties of recursively compact sets (cf. [GHR11, Propositions 1 & 3]; see also [He25, Proposition 3.23]).

Proposition 3.19. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. Assume that X is recursively compact, and $\{K_i\}_{i\in I}$ is uniformly recursively compact. Then the following statements are true:*

(i) *Let $x_i \in X$ for each $i \in I$. Then $\{x_i\}_{i\in I}$ is uniformly computable if and only if the sequence $\{\{x_i\}\}_{i\in I}$ of singletons is uniformly recursively compact.*
(ii) *$\{X \smallsetminus K_i\}_{i\in I}$ is uniformly lower semi-computable open.*
(iii) *If $\{U_i\}_{i\in I}$ is uniformly lower semi-computable open, then $\{K_i \smallsetminus U_i\}_{i\in I}$ is uniformly recursively compact.*
(iv) *If $\{f_i\}_{i\in I}$ is a sequence of uniformly lower (resp. upper) semi-computable functions $f_i\colon X \to \mathbb{R}$ with respect to $\{K_i\}_{i\in I}$, then $\{\inf_{x\in K_i} f_i(x)\}_{i\in I}$ (resp. $\{\sup_{x\in K_i} f_i(x)\}_{i\in I}$) is uniformly lower (resp. upper) semi-computable.*
(v) *If $\{T_i\}_{i\in I}$ is a sequence of uniformly computable functions $T_i\colon X \to X$ with respect to $\{K_i\}_{i\in I}$, then $\{T_i(K_i)\}_{i\in I}$ is uniformly recursively compact.*

Next, we investigate whether the property of uniform computability for recursively compact sets is preserved under the union and intersection.

Proposition 3.20. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space. Suppose X is recursively compact, H and L are two nonempty recursively enumerable sets with $L \subseteq I \times H$, and $\{K_{i,h}\}_{(i,h)\in L}$ is uniformly recursively compact. Denote $L_h \coloneqq \{(i,h) \in L : i \in I\}$ for each $h \in H$. Then the following statements are true:*

(i) *$\{\bigcap\{K_{i,h} : (i,h) \in L_h\}\}_{h\in H}$ is uniformly recursively compact.*
(ii) *If the function $F\colon H \to \mathbb{N}$ defined by $F(h) \coloneqq \operatorname{card} L_h$ for $h \in H$ is recursive, then $\{\bigcup\{K_{i,h} : (i,h) \in L_h\}\}_{h\in H}$ is uniformly recursively compact.*

Proposition 3.20 (i) follows immediately from Proposition 3.9 and Proposition 3.19 (ii) and (iii). Moreover, Proposition 3.20 (ii) follows from Definition 3.17. As a corollary of Proposition 3.20 (ii), we obtain the following result.

Moreover, given the recursive compactness of X, the computability of functions is preserved under a finite number of operations among additions and multiplications. We summarize this property in the following result (cf. [We00, Corollary 4.3.4]; see also [He25, Proposition 3.26]).

Proposition 3.21. *Let $(X,\,\rho,\,\mathcal{S})$ be a computable metric space in which X is recursively compact, and H be a nonempty recursively enumerable set. Assume that $\{f_i\}_{i\in I}$ (resp. $\{g_h\}_{h\in H}$) is a sequence of uniformly computable functions $f_i\colon X \to \mathbb{R}$ (resp. $g_h\colon X \to \mathbb{R}$). Then $\{f_i + g_h\}_{(i,h)\in I\times H}$, $\{f_i \cdot g_h\}_{(i,h)\in I\times H}$ are two sequences of uniformly computable functions.*

Next, we recall the definition of recursively precompact metric space.

Definition 3.22 (Recursively precompact metric space). Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_i\}_{i\in\mathbb{N}}$. Then $(X,\, \rho,\, \mathcal{S})$ is called *recursively precompact* if there exists an algorithm that, for each $n \in \mathbb{N}$, on input n, outputs a finite subset $\{r_i : 1 \leqslant i \leqslant m\}$ of $\mathbb{N}$ such that $X = \bigcup_{i=1}^m B(s_{r_i}, 2^{-n})$.

Finally, we record the following useful characterization of complete recursively precompact metric spaces (see e.g. [GHR11, Proposition 4]).

Proposition 3.23. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space. Then X is recursively compact if and only if (X, ρ) is complete and $(X,\, \rho,\, \mathcal{S})$ is recursively precompact.*

Computability of probability measures. Building upon the theory of computable functions and recursively compact sets, we now discuss the computability of probability measures. We begin by reviewing the computable structure on the measure space $\mathcal{P}(X)$ introduced in [HR09, Section 4] (cf. [HR09, Proposition 4.1.3]; see also [He25, Proposition 3.29]).

Proposition 3.24. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space with $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$. Assume that X is recursively compact in $(X,\, \rho,\, \mathcal{S})$. Then the following statements are true:*

(i) *There exists an enumeration $\mathcal{Q}_{\mathcal{S}} = \{\nu_k\}_{k\in\mathbb{N}}$ of the set of Borel probability measures that are supported on finitely many points in $\{s_n : n \in \mathbb{N}\}$ and assign rational values to them such that there exists an algorithm that, for each $k \in \mathbb{N}$, upon input k, outputs a sequence $\{n_l\}_{l=1}^p$ of integers and a sequence $\{q_l\}_{l=1}^p$ of positive rational numbers satisfying that $\sum_{l=1}^p q_l = 1$ and $\nu_k = \sum_{l=1}^p q_l \delta_{s_{n_l}}$.*

(ii) *$(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_{\mathcal{S}})$ is also a computable metric space in which $\mathcal{P}(X)$ is recursively compact, where W_ρ is the Wasserstein–Kantorovich metric on $\mathcal{P}(X)$ (see (2.2)).*

Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space and assume that X is recursively compact. We endow the measure space $\mathcal{P}(X)$ with the computable structure $(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_{\mathcal{S}})$ given by Proposition 3.24.

The computability of measures is then defined via Definition 3.5. Specifically, a sequence $\{\mu_i\}_{i\in I}$ of measures in $\mathcal{P}(X)$ is *a sequence of uniformly computable measures* if it is uniformly computable in $(\mathcal{P}(X),\, W_\rho,\, \mathcal{Q}_{\mathcal{S}})$, and a single measure $\mu \in \mathcal{P}(X)$ is *a computable measure* if the corresponding constant sequence consisting of μ is uniformly computable.

Finally, we prove an effective openness result by generalizing [Zi06, Theorem 18(d)] from Euclidean spaces to certain computable metric spaces.

Proposition 3.25. *Let $(X,\, \rho,\, \mathcal{S})$ be a computable metric space, where X is recursively compact and open balls are connected, and $T\colon X \to X$ be a computable function. Assume that I is a non-empty recursively enumerable set, and that $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets with the property that T is injective and open on U_i for each $i \in I$. Then $\{T(U_i)\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets.*

Proof. Let $\{q_m\}_{m\in\mathbb{N}}$ be an effective enumeration of $\mathbb{Q}^+$ and $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$. Since X is recursively compact, by Definition 3.17, X is compact. Hence, the diameter $\operatorname{diam}_\rho(X)$ of X is finite and $X = B\bigl(s_1, 2\operatorname{diam}_\rho(X)\bigr)$. Thus, by the hypotheses of Proposition 3.25, X is connected.

First, we consider the case where there is a sequence $\{x_i\}_{i\in I}$ of uniformly computable points and a sequence $\{r_i\}_{i\in I}$ of uniformly computable real numbers such that for each $i \in I$, we have $x_i \in \mathcal{S}$, $r_i \in \mathbb{Q}^+ \cup \{0\}$, $U_i = B(x_i, r_i)$, and that T is injective on $\overline{B}(x_i, r_i)$. Then $\{f_i\}_{i\in I}$, given by $f_i(x) \coloneqq \rho(x, x_i) - r_i$ for all $i \in I$ and $x \in X$, is a sequence of uniformly computable functions.

Write $S_i \coloneqq (f_i)^{-1}(\{0\})$, $\overline{B}_i \coloneqq \overline{B}(x_i, r_i)$, and $A_i \coloneqq X \smallsetminus \overline{B}_i$ for each $i \in I$. Since $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets, by Proposition 3.8, there exists a recursively enumerable set $E \subseteq \mathbb{N} \times I$ such that $\{s_n : (n, i) \in E\} = \{s_n : n \in \mathbb{N}\} \cap U_i$ for each $i \in I$. Consider an arbitrary $(n, i) \in E$ with $S_i \neq \emptyset$. Since $s_n \in U_i$ and T are injective on $\overline{B}_i = U_i \cup S_i$, we have that

$T(s_n) \notin T(S_i)$. Note that X is compact. Then for each $i \in I$, by the closeness of S_i and the continuity of T, we have that $T(S_i)$ is compact. Hence, $\rho(T(s_n), T(S_i)) > 0$.

Claim. Suppose $i \in I$. Then $S_i = \emptyset$ is equivalent to $X \subseteq U_i$. Moreover, if $S_i \neq \emptyset$, then we have $T(U_i) = \bigcup\{B(T(s_n), \rho(T(s_n), T(S_i))) : (n,i) \in E\}$; otherwise $T(U_i) = X$.

Proof of the claim. First, we consider $i \in I$ with $r_i = 0$. Then $S_i = \{x_i\}$ and $U_i = \emptyset$. Thus, $\{n \in \mathbb{N} : (n,i) \in E\} = \emptyset$. Hence, $T(U_i) = \emptyset = \bigcup\{B(T(s_n), \rho(T(s_n), T(S_i))) : (n,i) \in E\}$.

Now we consider $i \in I$ with $r_i \in \mathbb{Q}^+$. Then $U_i \neq \emptyset$. Indeed, the equivalence between $S_i = \emptyset$ and $X \subseteq U_i$ follows from the openness of U_i and A_i, the connectedness of X, and $X = U_i \cup S_i \cup A_i$.

Then we assume that $S_i \neq \emptyset$ and prove that $T(U_i) = \bigcup\{B(T(s_n), \rho(T(s_n), T(S_i))) : (n,i) \in E\}$. Denote $B_{n,i} \coloneqq B(T(s_n), \rho(T(s_n), T(S_i)))$ for each $n \in \mathbb{N}$ with $(n,i) \in E$. Fix an arbitrary $n \in \mathbb{N}$ with $(n,i) \in E$. Since T is continuous on X, $T(\overline{B}_i)$ is a closed set. Thus $T(\overline{B}_i) = \overline{T(\overline{B}_i)} \supseteq \overline{T(U_i)} \supseteq \partial(T(U_i))$. Note that T is open on U_i and U_i is open. Then $T(U_i)$ is also open. Thus, $\partial(T(U_i)) \cap T(U_i) = \emptyset$. Hence $\partial(T(U_i)) \subseteq T(S_i)$. Moreover, by the hypotheses of Proposition 3.25, $B_{n,i}$ is connected. By construction, we have $B_{n,i} \cap T(S_i) = \emptyset$. Combined with $\partial(T(U_i)) \subseteq T(S_i)$, this implies that $B_{n,i} \cap \partial(T(U_i)) = \emptyset$. By $s_n \in U_i$, we have $T(s_n) \in B_{n,i} \cap T(U_i)$. By the connectedness of $B_{n,i}$, it follows from $B_{n,i} \cap \partial(T(U_i)) = \emptyset$ that $B_{n,i} \subseteq T(U_i)$.

Now we establish that for each $x \in U_i$, there exists $n \in \mathbb{N}$ with $T(x) \in B_{n,i}$. Since $T(U_i)$ is open and $T(x) \in T(U_i)$, there exists $r_0 > 0$ with $B(T(x), r_0) \subseteq T(U_i)$. Note that T is injective on $\overline{B}_i = U_i \cup S_i$. Then $T(U_i) \cap T(S_i) = \emptyset$, and thus, $B(T(x), r_0) \cap T(S_i) = \emptyset$. Since $\{s_n : n \in \mathbb{N}\}$ is dense in X, and $\{s_n : (n,i) \in E\} = \{s_n : n \in \mathbb{N}\} \cap U_i$, it follows from the openness of U_i that $\{s_n : (n,i) \in E\}$ is dense in U_i. Hence, since T is injective and open on U_i, $\{T(s_n) : (n,i) \in E\}$ is dense in $T(U_i)$. Thus, there exists $m \in \mathbb{N}$ such that $(m,i) \in E$ and $\rho(T(s_m), T(x)) < r_0/2$. Hence, we obtain that $T(x) \in B(T(s_m), r_0/2) \subseteq B(T(x), r_0)$. Then we argue that $\rho(T(s_m), T(S_i)) \geqslant r_0/2$ by contradiction. Otherwise, we have $T(S_i) \cap B(T(s_m), r_0/2) \neq \emptyset$, which leads to a contradiction, since $B(T(x), r_0) \cap T(S_i) = \emptyset$ and $B(T(s_m), r_0/2) \subseteq B(T(x), r_0)$. So far we have shown that $\rho(T(s_m), T(S_i)) \geqslant r_0/2$. Thus, by the definition of $B_{m,i}$, we have $T(x) \in B(T(s_m), r_0/2) \subseteq B_{m,i}$.

Finally, we assume that $S_i = \emptyset$. Since T is continuous on X and $U_i = \overline{B}_i$ is compact, $T(U_i)$ is also compact. Hence, since $T(U_i)$ is open, by the connectedness of X, we obtain $T(U_i) = X$. We have completed the proof of the claim.

Now we prove the original statement. By Definition 3.7, $\mathbb{R} \setminus \{0\}$ is a lower semi-computable open set. Hence, since $\{f_i\}_{i\in I}$ is a sequence of uniformly computable functions, by Proposition 3.14, we obtain that $\{(f_i)^{-1}(\mathbb{R} \setminus \{0\})\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets in X. Combining this with Proposition 3.19 (iii) and the fact that $S_i = X \setminus (f_i)^{-1}(\mathbb{R} \setminus \{0\})$, we obtain that $\{S_i\}_{i\in I}$ is a sequence of uniformly recursively compact sets. By Proposition 3.19 (v), it follows from the computability of T that $\{T(S_i)\}_{i\in I}$ is a sequence of uniformly recursively compact sets. Since $\{s_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly computable points, by Definitions 3.5 and 3.12, $\{T(s_n)\}_{n\in\mathbb{N}}$ is a sequence of uniformly computable points. Hence, $\{U_{n,m}\}_{n,m\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable open sets, where $U_{n,m} \coloneqq \{x \in X : \rho(x, T(s_n)) > q_m\}$ for all $n, m \in \mathbb{N}$. By Proposition 3.18, there exists an algorithm $\mathcal{A}(n,m,i)$ which for all $n, m \in \mathbb{N}$, and $i \in I$, on input n, m, and i, halts if and only if $T(S_i) \subseteq U_{n,m}$.

Define the set $L \subseteq \mathbb{N}^2 \times I$ by $L \coloneqq \{(n,m,i) \in \mathbb{N}^2 \times I : (n,i) \in E, \text{ and } \mathcal{A}(n,m,i) \text{ halts}\}$. By Definition 3.1, we obtain that L is a recursively enumerable set. Note that $\{T(s_n)\}_{n\in\mathbb{N}}$ is a sequence of uniformly computable points and that $\{q_m\}_{m\in\mathbb{N}}$ is an effective enumeration of $\mathbb{Q}^+$. Then by Proposition 3.10, $\{B(T(s_n), q_m) : (n,m,i) \in L\}$ is a sequence of uniformly lower semi-computable open sets. Thus, it follows from Proposition 3.9 that $\{V_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets, where $V_i \coloneqq \bigcup\{B(T(s_n), q_m) : (n,m,i) \in L\}$ for each $i \in I$.

Next, we apply the claim to show that $V_i = T(U_i)$ for each $i \in I$. Indeed, by the definition of $\{U_{n,m}\}_{n,m\in\mathbb{N}}$, $\mathcal{A}(n,m,i)$ halts if and only if $\rho(T(S_i), T(s_n)) > q_m$ for all $n, m \in \mathbb{N}$ and $i \in I$. Now we consider $i \in I$ with $S_i \neq \emptyset$. Then by the definition of $\{V_i\}_{i\in I}$ and $\{B_{n,i}\}_{(n,i)\in E}$, it follows from the claim that $V_i = \bigcup\{B(T(s_n), q_m) : (n,m,i) \in L\} = \bigcup_{(n,i)\in E} \bigcup\{B(T(s_n), q_m) : \mathcal{A}(n,m,i) \text{ halts}\} = \bigcup_{(n,i)\in E} B_{n,i} = T(U_i)$. We turn to consider $i \in I$ with $S_i = \emptyset$. Thus, $T(S_i) \subseteq U_{n,m}$, namely, $\mathcal{A}(n,m,i)$ halts for all $n, m \in \mathbb{N}$. Hence, by the definition of $\{V_i\}_{i\in I}$ and $\operatorname{diam}_\rho(X) < +\infty$, it follows from the claim that $V_i = \bigcup\{B(T(s_n), q_m) : (n,m,i) \in L\} = \bigcup\{B(T(s_n), q_m) : n, m \in \mathbb{N}, \text{ and } (n,i) \in E\} = X = T(U_i)$.

Thus, $\{T(U_i)\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets.

Finally, we establish the general case. Let $\{D_n\}_{n\in\mathbb{N}}$ be an effective enumeration of extended ideal balls in $(X, \rho, \mathcal{S})$. Hence, there exist three recursive functions $f\colon \mathbb{N} \to \mathbb{N}$, $u\colon \mathbb{N} \to \mathbb{N}$, and $v\colon \mathbb{N} \to \mathbb{N}$ such that $D_n = B\bigl(s_{f(n)}, \frac{u(n)-1}{v(n)}\bigr)$ for each $n \in \mathbb{N}$. Since $\{U_i\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets, by Definition 3.7, there exists a recursive function $g\colon \mathbb{N} \times I \to \mathbb{N}$ such that $U_i = \bigcup_{n\in\mathbb{N}} D_{g(n,i)}$ for each $i \in I$. Now we define $x_{n,m,i} \coloneqq s_{f(g(n,i))}$ and $r_{n,m,i} \coloneqq \frac{(u(g(n,i))-1)m}{v(g(n,i))(m+1)}$ for all $n, m \in \mathbb{N}$, and $i \in I$. Then $\{x_{n,m,i}\}_{n,m\in\mathbb{N},i\in I}$ is a sequence of uniformly computable points and $\{r_{n,m,i}\}_{n,m\in\mathbb{N},i\in I}$ is a sequence of uniformly computable real numbers. Note that $x_{n,m,i} \in \mathcal{S}$ and $r_{n,m,i} \in \mathbb{Q}^+ \cup \{0\}$. By the hypotheses in Proposition 3.25, T is injective and open on U_i for each $i \in I$. Then T is injective on $\overline{B}(x_{n,m,i}, r_{n,m,i})$ and T is open on $B(x_{n,m,i}, r_{n,m,i})$ for all $n, m \in \mathbb{N}$, and $i \in I$. Hence, by the discussion in the first case above, we obtain that $\{T(B(x_{n,m,i}, r_{n,m,i}))\}_{n,m\in\mathbb{N},i\in I}$ is a sequence of uniformly lower semi-computable open sets. Note that by the constructions, we have $T(U_i) = \bigcup_{n\in\mathbb{N}} T\bigl(D_{g(n,i)}\bigr) = \bigcup_{n,m\in\mathbb{N}} T(B(x_{n,m,i}, r_{n,m,i}))$ for each $i \in I$. Therefore, by Proposition 3.9, $\{T(U_i)\}_{i\in I}$ is a sequence of uniformly lower semi-computable open sets. □

3.2. **Thermodynamic formalism.** We review basic concepts from ergodic theory. For more detailed discussions, we refer the reader to [Wa82, Section 4].

Let $(X, \mathcal{B}, \mu)$ be a probability space. A *partition* $\xi = \{A_h : h \in H\}$ of $(X, \mathcal{B}, \mu)$ is a disjoint collection of elements of $\mathcal{B}$ whose union is X, where H is a countable index set. For each pair of partitions $\xi = \{A_h : h \in H\}$ and $\eta = \{B_l : l \in L\}$ of X, their *join* is the partition $\xi \vee \eta \coloneqq \{A_h \cap B_l : h \in H,\ l \in L\}$.

Assume that $T\colon X \to X$ is a measure-preserving transformation of $(X, \mathcal{B}, \mu)$. Consider a partition $\xi = \{A_h : h \in H\}$ of X. For each $n \in \mathbb{N}$, $T^{-n}(\xi)$ denotes the partition $\bigl\{T^{-1}(A_h) : h \in H\bigr\}$, and ξ_T^n denotes the join $\xi \vee T^{-1}(\xi) \vee \cdots \vee T^{-(n-1)}(\xi)$. The *entropy* of ξ is $H_\mu(\xi) \coloneqq -\sum_{h\in H} \mu(A_h) \log(\mu(A_h)) \in [0, +\infty]$, where $0 \log 0$ is defined to be zero. One can show that if $H_\mu(\xi) < +\infty$, then $\lim\limits_{n\to+\infty} H_\mu(\xi_T^n)/n$ exists (see e.g. [Wa82, Chapter 4]). We denote this limit by $h_\mu(T, \xi)$ and call it the *measure-theoretic entropy of T relative to ξ*. The *measure-theoretic entropy* of T for μ is defined as

$$h_\mu(T) \coloneqq \sup\{h_\mu(T, \xi) : \xi \text{ is a partition of } X \text{ with } H_\mu(\xi) < +\infty\}. \tag{3.1}$$

We now introduce thermodynamic formalism, a particular branch of ergodic theory. The main objects of study are the topological pressure and equilibrium states (see e.g. [PU10, Wa82]; for the general Borel-measurable setting used in Approach II, see e.g. [IT10, Definition 1.1], [DeT17, Section 2.3], and [DoT23, Chapter 1.4]).

Let (X, ρ) be a compact metric space, $T\colon X \to X$ be a Borel-measurable transformation such that $\mathcal{M}(X, T) \neq \emptyset$, and $\phi\colon X \to [-\infty, +\infty]$ be a Borel function. Then the *topological pressure* of the potential ϕ with respect to the transformation T is given by

$$P(T, \phi) \coloneqq \sup\bigl\{h_\mu(T) + \langle \mu, \phi \rangle : \mu \in \mathcal{M}(X, T) \text{ and } \langle \mu, \phi \rangle > -\infty\bigr\}. \tag{3.2}$$

A measure $\mu \in \mathcal{M}(X, T)$ that attains the supremum in (3.2) is called an *equilibrium state* for the transformation T and the potential ϕ. Denote the set of all such measures by $\mathcal{E}(T, \phi)$. In particular, when the potential ϕ is the constant function 0, we denote $h_{\text{top}}(T) \coloneqq P(T, 0)$ and say that a measure $\mu \in \mathcal{M}(X, T)$ is a *measure of maximal entropy* of T if $\mu \in \mathcal{E}(T, 0)$.

4. Proof of Main Results

4.1. **Cones and projective metrics.** First, we introduce some notations in the cone technique. Let E be a vector space over $\mathbb{R}$. A *convex cone* in E is a subset $\mathcal{C} \subseteq E \setminus \{0\}$ satisfying the following properties:

(i) $tu \in \mathcal{C}$ for all $u \in \mathcal{C}$ and $t > 0$.
(ii) $\lambda u + \eta v \in \mathcal{C}$ for all u, $v \in \mathcal{C}$ and λ, $\eta > 0$.
(iii) $\{u \in E : u \in \overline{\mathcal{C}},\ -u \in \overline{\mathcal{C}}\} = \{0\}$.

Here $\overline{\mathcal{C}}$ is the set consisting of $u \in E$ satisfying that there exists $v \in \mathcal{C}$ and $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ satisfying that $u + t_n v \in \mathcal{C}$ for each $n \in \mathbb{N}$ and that t_n converges to 0 as n tends to $+\infty$.

Let $\mathcal{C}$ be a convex cone. For each pair of $u, v \in \mathcal{C}$, we define

$$A(u,v) \coloneqq \sup\{t > 0 \colon v - tu \in \mathcal{C}\} \text{ and } B(u,v) \coloneqq \inf\{s > 0 \colon su - v \in \mathcal{C}\},$$

with the convention $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$, where $\emptyset$ denotes the empty set. We have that $A(u,v)$ is finite, $B(u,v)$ is positive and $A(u,v) \leqslant B(u,v)$ for all $u, v \in \mathcal{C}$. Define

$$\Theta(u,v) \coloneqq \log\left(\frac{B(u,v)}{A(u,v)}\right),$$

with $\Theta(u,v)$ possibly infinity in the case $A(u,v) = 0$ or $B(u,v) = +\infty$. Note that $\Theta(u,v)$ is well-defined and takes values in $[0,+\infty]$. Since $\Theta(u,v) = 0$ if and only if $u = tv$ for some $t > 0$, Θ defines a pseudo-metric on $\mathcal{C}$. In this way, Θ induces a metric on a projective quotient space of $\mathcal{C}$ called the *projective metric of* $\mathcal{C}$.

Let E_1, E_2 be vector spaces over $\mathbb{R}$, if $L \colon E_1 \to E_2$ is a linear operator, and $\mathcal{C}_1, \mathcal{C}_2$ are convex cones in E_1, E_2, respectively, such that $L(\mathcal{C}_1) \subset \mathcal{C}_2$, then $\Theta_2(L(u), L(v)) \leqslant \Theta_1(u,v)$ for all $u, v \in \mathcal{C}_1$, where Θ_1 and Θ_2 are the projective metrics in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

In general, L need not be a strict contraction, that will be the case for instance if $L(\mathcal{C}_1)$ had finite diameter in $\mathcal{C}_2$ according to the next result.

Proposition 4.1. *Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be a convex cone in a vector space E_1 (resp. E_2) with the projective metric Θ_1 (resp. Θ_2), and $L \colon E_1 \to E_2$ be a linear operator with $L(\mathcal{C}_1) \subset \mathcal{C}_2$. Assume that $\Delta \coloneqq \operatorname{diam}_{\Theta_2}(L(\mathcal{C}_1)) < +\infty$, then*

$$\Theta_2(L(u), L(v)) \leqslant \left(1 - e^{-\Delta}\right) \cdot \Theta_1(u,v) \quad \text{for all } u,\ v \in \mathcal{C}_1.$$

The Ruelle-Perron-Frobenius operator $\mathcal{L}$ associated to $f : M \to M$ and $\phi : M \to \mathbb{R}$ is the linear operator defined by

$$\mathcal{L}(u)(x) = \sum_{y \in f^{-1}(x)} u(y)\exp(\phi(y)) \quad \text{for } u \in C(M) \text{ and } x \in M. \tag{4.1}$$

Now we summarize Lemmas 4.1, 6.5, and Theorem B in [VV10] here in our context.

Theorem 4.2. *Under the Assumptions and the Additional Assumptions, there exists $\nu \in \mathcal{P}(X)$ with $\mathcal{L}^*(\nu) = \exp(P(f,\phi)) \cdot \nu$.*

Given $\alpha \in (0,1)$ and $\delta > 0$, we say that a function $u : M \to \mathbb{R}$ is (C,α)-*Hölder continuous in balls of radius* δ if there exists $C > 0$ satisfying that

$$|u(x) - u(y)| \leqslant Cd(x,y)^\alpha \quad \text{for all } x,\ y \in M \text{ with } d(x,y) < \delta. \tag{4.2}$$

Denote by $|u|_{\alpha,\delta}$ the smallest constant C satisfying (4.2).

Lemma 4.3. *There exists a computable function $F \colon (1,+\infty) \times (0,1) \to \mathbb{R}$ such that for each $\delta > 0$, if $u \colon M \to \mathbb{R}$ is (C,α)-Hölder continuous in balls of radius δ, then u is $(C(F(l,\alpha)),\alpha)$-Hölder continuous in balls of radius $l\delta$. In particular, $|u|_{\alpha,l\delta} \leqslant F(l,\alpha)|u|_{\alpha,\delta}$.*

Remark 4.4. It follows from [CV13, Lemma 3.5] that the above result holds in the case where $F(l,\alpha) = 1 + (l-1)^{\alpha}$, $l \in (1,2)$, and $\alpha > 0$.

For all $\alpha \in (0,1)$, $\delta > 0$, and $k > 0$, we define

$$\mathcal{C}_{\alpha,\delta,k} := \Big\{ u \in C(M) : u > 0 \text{ and } |u|_{\alpha,\delta} \leqslant k \cdot \inf_{x \in M} u(x) \Big\}. \tag{4.3}$$

For the cone $\mathcal{C}_{\alpha,\delta,k}$, an explicit expression of its projective metric $\Theta_{\alpha,\delta,k}$ is given in the following lemma (see [CV13, Lemma 4.2]).

Lemma 4.5. *For all $\alpha \in (0,1)$, $\delta > 0$, and $k > 0$, we have*

$$\Theta_{\alpha,\delta,k}(u,v) = \log\biggl(\frac{B_{\alpha,\delta,k}(u,v)}{A_{\alpha,\delta,k}(u,v)} \biggr) \quad \textit{for all } u,\, v \in C_{\alpha,\delta,k},$$

where

$$A_{\alpha,\delta,k}(u,v) := \inf_{d(x,y)<\delta, z \in M} \frac{kd(x,y)^{\alpha} v(z) - (v(x) - v(y))}{kd(x,y)^{\alpha} u(z) - (u(x) - u(y))}$$

and

$$B_{\alpha,\delta,k}(u,v) := \sup_{d(x,y)<\delta, z \in M} \frac{kd(x,y)^{\alpha} v(z) - (v(x) - v(y))}{kd(x,y)^{\alpha} u(z) - (u(x) - u(y))}.$$

In particular, we have

$$A_{\alpha,\delta,k}(u,v) \leqslant \inf_{z \in M} \biggl\{ \frac{v(z)}{u(z)} \biggr\} \quad \textit{and} \quad B_{\alpha,\delta,k}(u,v) \geqslant \sup_{z \in M} \biggl\{ \frac{v(z)}{u(z)} \biggr\}.$$

Proposition 4.6. *Given $\alpha \in (0,1)$ and $k > 0$. Suppose $\delta > 0$ and that there exists a sequence $\{x_i\}_{i=1}^{m-1}$ of points in M such that $\bigcup_{i=1}^{m-1} B(x_i, \delta/3) = M$. Then we have*

$$\sup_{z \in M} u(z) \leqslant (1 + m \cdot k \cdot \operatorname{diam}(M)^{\alpha}) \cdot \inf_{z \in M} u(z) \quad \textit{for each } u \in \mathcal{C}_{\alpha,\delta,k}.$$

Proof. By $u \in \mathcal{C}_{\alpha,\delta,k}$, we have $|u|_{\alpha,\delta} \leqslant k \cdot \inf_{x \in M} u(x)$. Since M is compact, there exist two points y and z in M with $u(y) = \sup_{x \in M} u(x)$ and $u(z) = \inf_{x \in M} u(x)$. Now we define a graph $G = (V, E)$ by $V = \{x_i : 1 \leqslant i \leqslant m-1\}$ and $E = \{(x_i, x_k) : 1 \leqslant i, k \leqslant m-1 \text{ and } d(x_i, x_k) < \delta\}$. It follows from the connectedness of M and $\bigcup_{i=1}^{m-1} B(x_i, \delta/3) = M$ that G is a connected graph. Hence, there exists an integer $2 \leqslant s \leqslant m+1$ and a sequence $\{p_k\}_{k=1}^{s}$ of points such that $p_1 = y$, $p_s = z$, and $d(p_k, p_{k+1}) < \delta$ for each integer $1 \leqslant k \leqslant s-1$. Thus, we obtain that

$$|u(y) - u(z)| \leqslant \sum_{k=1}^{s-1} |u(p_{k+1}) - u(p_k)| \leqslant \sum_{k=1}^{s-1} |u|_{\alpha,\delta} \cdot d(p_k, p_{k+1})^{\alpha} \leqslant (s-1)|u|_{\alpha,\delta} \cdot [\operatorname{diam}(M)]^{\alpha}.$$

Therefore, we have

$$\sup_{x \in M} u(x) \leqslant \inf_{x \in M} u(x) + (s-1)|u|_{\alpha,\delta} \cdot \operatorname{diam}(M)^{\alpha} \leqslant \inf_{x \in M} u(x)(1 + m \cdot k \cdot \operatorname{diam}(M)^{\alpha}).$$

□

Next, we show that for each $\lambda \in (0,1)$, the cone $\mathcal{C}_{\alpha,\delta,\lambda k}$ has a finite $\Theta_{\alpha,\delta,k}$-diameter.

Proposition 4.7. *Given $\alpha \in (0,1)$, $\delta > 0$, $k > 0$, and $\lambda \in (0,1)$. Then $\mathcal{C}_{\alpha,\delta,\lambda k} \subseteq \mathcal{C}_{\alpha,\delta,k}$. Moreover, assume that there exists a family of $(m-1)$ balls of radius $\delta/3$ that covers M. Then we have*

$$\operatorname{diam}_{\Theta_{\alpha,\delta,k}}(\mathcal{C}_{\alpha,\delta,\lambda k}) \leqslant \Delta(\alpha,\delta,k,\lambda), \tag{4.4}$$

where

$$\Delta(\alpha,\delta,k,\lambda) := 2\log\biggl(\frac{1 + m \cdot \lambda \cdot k \cdot diam(M)^{\alpha} + \lambda}{1 - \lambda} \biggr). \tag{4.5}$$

Proof. By (4.3) and $\lambda \in (0,1)$, we have $\mathcal{C}_{\alpha,\delta,\lambda k} \subseteq \mathcal{C}_{\alpha,\delta,k}$. Let $u, v \in \mathcal{C}_{\alpha,\delta,\lambda k}$. By Lemma 4.5, we have

$$\Theta_{\alpha,\delta,k}(u,v) \leqslant \log\left(\frac{k \cdot \sup_{x\in M} v(x) + \lambda \cdot k \cdot \inf_{x\in M} v(x)}{k \cdot \inf_{x\in M} v(x) - \lambda \cdot k \cdot \inf_{x\in M} v(x)} \cdot \frac{k \cdot \sup_{x\in M} u(x) + \lambda \cdot k \cdot \inf_{x\in M} u(x)}{k \cdot \inf_{x\in M} u(x) - \lambda \cdot k \cdot \inf_{x\in M} u(x)}\right).$$

Combining this with Proposition 4.6, we obtain that

$$\begin{aligned}\Theta_{\alpha,\delta,k}(u,v) &\leqslant \log \frac{(1 + m \cdot \lambda \cdot k \cdot \operatorname{diam}(M)^\alpha + \lambda) \inf_{x\in M} u(x)}{(1-\lambda) \inf_{x\in M} u(x)} \\ &\quad + \log \frac{(1 + m \cdot \lambda \cdot k \cdot \operatorname{diam}(M)^\alpha + \lambda) \inf_{x\in M} v(x)}{(1-\lambda) \inf_{x\in M} v(x)} \\ &\leqslant 2\log\left(\frac{1 + m \cdot \lambda \cdot k \cdot \operatorname{diam}(M)^\alpha + \lambda}{1-\lambda}\right) = \Delta(\alpha,\delta,k,\lambda).\end{aligned}$$

□

Theorem 4.8. *Given $\alpha \in (0,1)$, and $\delta > 0$. Under the Assumptions and the Additional Assumptions, we have $\mathcal{L}(\mathcal{C}_{\alpha,\delta,k}) \subseteq \mathcal{C}_{\alpha,\delta,\lambda k}$ for each $k \geqslant (m \cdot \operatorname{diam}(M)^\alpha)^{-1}$, where*

$$\lambda := \frac{\big((\deg(f) - q)\sigma^{-\alpha} + qL^\alpha F(L,\alpha)\big) \cdot \exp\big(\varepsilon_\phi\big)}{\deg(f)} + 2mL^\alpha \varepsilon_\phi \cdot \operatorname{diam}(M)^\alpha. \tag{4.6}$$

Proof. Take $u \in \mathcal{C}_{\alpha,\delta,k}$ and check that $\mathcal{L}(u) \in \mathcal{C}_{\alpha,\delta,\lambda k}$. Indeed, by (4.3), we have for each $x \in M$,

$$\mathcal{L}(u)(x) = \sum_{y\in f^{-1}(x)} u(y)\exp(\phi(y)) \geqslant \deg(f) \cdot \inf_{z\in M} u(z) \cdot \exp\big(\inf_{z\in M} \phi(z)\big) > 0. \tag{4.7}$$

Thus, it suffices to show that $\mathcal{L}(u) \in C(M)$ and $|\mathcal{L}(u)|_{\alpha,\delta} \leqslant \lambda k \inf_{z\in M}(\mathcal{L}(u)(z))$.

Now we consider an arbitrary pair of distinct points x, $y \in M$ with $d(x,y) < \delta$. Since δ is acceptable, there exists a sequence $\{(x_i,y_i)\}_{i=1}^{\deg(f)}$ of pairs of paired preimages associated to x and y satisfying that

$$\frac{d(x_i,y_i)}{d(x,y)} \leqslant \begin{cases} \sigma^{-1}, & 1 \leqslant i \leqslant \deg(f) - q \\ L, & \textit{otherwise} \end{cases} \quad \text{for each integer } 1 \leqslant i \leqslant \deg(f). \tag{4.8}$$

One can see that

$$\begin{aligned}|\mathcal{L}(u)(x) - \mathcal{L}(u)(y)| &\leqslant \sum_{i=1}^{\deg(f)} |u(x_i)\exp(\phi(x_i)) - u(y_i)\exp(\phi(y_i))| \\ &\leqslant \sum_{i=1}^{\deg(f)-q} |u(x_i) - u(y_i)||\exp(\phi(x_i))| \qquad (4.9) \\ &\quad + \sum_{i=\deg(f)-q+1}^{\deg(f)} |u(x_i) - u(y_i)||\exp(\phi(x_i))| \qquad (4.10) \\ &\quad + \sum_{i=1}^{\deg(f)} |u(y_i)||\exp(\phi(x_i)) - \exp(\phi(y_i))|. \qquad (4.11)\end{aligned}$$

Now we estimate (4.9), (4.10), and (4.11), respectively.

For (4.9): By (4.8), we have $d(x_i, y_i) \leqslant \sigma^{-1} d(x, y) < \delta$ for each integer $1 \leqslant i \leqslant \deg(f) - q$. Hence, since $u \in \mathcal{C}_{\alpha,\delta,k}$ and $\sup_{z \in M} \phi(z) - \inf_{z \in M} \phi(z) < \varepsilon_\phi$, by (4.3), we obtain that

$$
\begin{aligned}
(4.9) &\leqslant \exp\bigl(\sup_{z\in M} \phi(z)\bigr) \cdot \Biggl(\sum_{i=1}^{\deg(f)-q} |u|_{\alpha,\delta} \cdot d(x_i, y_i)^\alpha \Biggr) \\
&\leqslant (\deg(f) - q) \cdot \exp\bigl(\sup_{z\in M} \phi(z)\bigr) \cdot |u|_{\alpha,\delta} \cdot \sigma^{-\alpha} \cdot d(x,y)^\alpha \\
&\leqslant (\deg(f) - q) \cdot \exp\bigl(\varepsilon_\phi\bigr) \cdot \exp\bigl(\inf_{z\in M} \phi(z)\bigr) \cdot k \cdot \inf_{z\in M} u(z) \cdot \sigma^{-\alpha} \cdot d(x,y)^\alpha \\
&= (\deg(f) - q)\sigma^{-\alpha} \cdot \exp\bigl(\varepsilon_\phi\bigr) \cdot k \cdot \inf_{z\in M} u(z) \cdot \exp\bigl(\inf_{z\in M} \phi(z)\bigr) \cdot d(x,y)^\alpha.
\end{aligned}
\tag{4.12}
$$

For (4.10): By (4.8), we have $d(x_i, y_i) \leqslant L d(x, y) < L\delta$ for each integer $\deg(f) - q + 1 \leqslant i \leqslant \deg(f)$. Hence, since $u \in \mathcal{C}_{\alpha,\delta,k}$ and $\sup_{z \in M} \phi(z) - \inf_{z \in M} \phi(z) < \varepsilon_\phi$, by (4.3) and Lemma 4.3, we obtain that

$$
\begin{aligned}
(4.10) &\leqslant \exp\bigl(\sup_{z\in M} \phi(z)\bigr) \cdot \Biggl(\sum_{i=\deg(f)-q+1}^{\deg(f)} |u|_{\alpha,L\delta} \cdot d(x_i, y_i)^\alpha \Biggr) \\
&\leqslant q \cdot \exp\bigl(\sup_{z\in M} \phi(z)\bigr) \cdot F(L,\alpha) \cdot |u|_{\alpha,\delta} \cdot L^\alpha \cdot d(x,y)^\alpha \\
&\leqslant q \cdot \exp\bigl(\varepsilon_\phi\bigr) \cdot \exp\bigl(\inf_{z\in M} \phi(z)\bigr) \cdot F(L,\alpha) \cdot k \cdot \inf_{z\in M} u(z) \cdot L^\alpha \cdot d(x,y)^\alpha \\
&= qL^\alpha F(L,\alpha) \cdot \exp\bigl(\varepsilon_\phi\bigr) \cdot k \cdot \inf_{z\in M} u(z) \cdot \exp\bigl(\inf_{z\in M} \phi(z)\bigr) \cdot d(x,y)^\alpha.
\end{aligned}
\tag{4.13}
$$

For (4.11): By (4.8) and $\sigma^{-1} < 1 < L$, we have $d(x_i, y_i) \leqslant L d(x, y)$ for each integer $1 \leqslant i \leqslant \deg(f)$. Hence, since $u \in \mathcal{C}_{\alpha,\delta,k}$, $k \geqslant (m \cdot \operatorname{diam}(M)^\alpha)^{-1}$, and $|\exp(\phi)|_\alpha < \varepsilon_\phi \exp(\inf_{z \in M} \phi(z))$, by (4.3) and Proposition 4.6, we obtain that

$$
\begin{aligned}
(4.11) &\leqslant \sup_{z\in M} u(z) \cdot \Biggl(\sum_{i=1}^{\deg(f)} |\exp(\phi)|_\alpha \cdot d(x_i, y_i)^\alpha \Biggr) \cdot \\
&\leqslant \deg(f) \cdot \sup_{z\in M} u(z) \cdot |\exp(\phi)|_\alpha \cdot L^\alpha \cdot d(x,y)^\alpha \\
&\leqslant \deg(f) \cdot (1 + m \cdot k \cdot \operatorname{diam}(M)^\alpha) \cdot \inf_{z\in M} u(z) \cdot \varepsilon_\phi \cdot \exp\bigl(\inf_{z\in M} \phi(z)\bigr) \cdot L^\alpha \cdot d(x,y)^\alpha \\
&\leqslant 2m \deg(f) L^\alpha \varepsilon_\phi \cdot \operatorname{diam}(M)^\alpha \cdot k \cdot \inf_{z\in M} u(z) \cdot \exp\bigl(\inf_{z\in M} \phi(z)\bigr) \cdot d(x,y)^\alpha.
\end{aligned}
\tag{4.14}
$$

Thus, by (4.12), (4.13), (4.14), and (4.6), we obtain that

$$
\begin{aligned}
\frac{|\mathcal{L}(u)(x) - \mathcal{L}(u)(y)|}{k \cdot \inf_{z\in M} u(z) \cdot \exp\bigl(\inf_{z\in M} \phi(z)\bigr) \cdot d(x,y)^\alpha} \leqslant &\bigl((\deg(f) - q)\sigma^{-\alpha} + qL^\alpha F(L,\alpha)\bigr) \cdot \exp\bigl(\varepsilon_\phi\bigr) \\
&+ 2m \deg(f) L^\alpha \varepsilon_\phi \cdot \operatorname{diam}(M)^\alpha = \lambda \deg(f).
\end{aligned}
$$

Combined with (4.7), this implies that

$$
\frac{|\mathcal{L}(u)(x) - \mathcal{L}(u)(y)|}{\inf_{z\in M}(\mathcal{L}(u)(z)) \cdot d(x,y)^\alpha} \leqslant \frac{|\mathcal{L}(u)(x) - \mathcal{L}(u)(y)|}{\deg(f) \cdot \inf_{z\in M} u(z) \cdot \exp\bigl(\inf_{z\in M} \phi(z)\bigr) \cdot d(x,y)^\alpha} \leqslant \lambda k.
$$

Therefore, we obtain that $\mathcal{L}(u) \in C(M)$ and $|\mathcal{L}(u)|_{\alpha,\delta} \leqslant \lambda k \inf_{z \in M}(\mathcal{L}(u)(z))$. □

Theorem 4.9. *Given $\alpha \in (0, 1)$, and $\delta > 0$. Under the Assumptions and the Additional Assumptions, in addition we define $\mathcal{L}_0(u) := \exp(-P(T, \phi))\mathcal{L}(u)$ for each $u \in C(M)$. Then the sequence $\{\mathcal{L}_0^n(\mathbb{1})\}_{n\in\mathbb{N}}$ of functions converges to a function h such that $\mathcal{L}_0(h) = h$ and*

$$
\|\mathcal{L}_0^{n+1}(\mathbb{1}) - h\|_\infty \leqslant 3(1 + m\lambda k \cdot \operatorname{diam}(M)^\alpha) \cdot (1 - \exp(-\Delta))^n \cdot \Delta
$$

for each $k \geqslant (m \cdot \operatorname{diam}(M)^\alpha)^{-1}$ and each integer $n \geqslant \frac{-\log(\Delta)}{\log(1-\exp(-\Delta))}$, where $\Delta := \Delta(\alpha, \delta, k, \lambda)$.

Proof. Set $h_n := \mathcal{L}_0^n(\mathbb{1})$ for each $n \in \mathbb{N}_0$. By Theorem 4.8, we have $\mathcal{L}(\mathcal{C}_{\alpha,\delta,k}) \subseteq \mathcal{C}_{\alpha,\delta,\lambda k}$. Hence, by definition, we obtain that $\mathcal{L}_0(\mathcal{C}_{\alpha,\delta,k}) \subseteq \mathcal{C}_{\alpha,\delta,\lambda k}$ and $h_n \in \mathcal{C}_{\alpha,\delta,\lambda k}$ for each $n \in \mathbb{N}$. Then by Proposition 4.1 for the operator $\mathcal{L}_0 \colon \mathcal{C}_{\alpha,\delta,k} \to \mathcal{C}_{\alpha,\delta,k}$, it follows by induction that

$$\Theta_{\alpha,\delta,k}(h_i, h_l) \leqslant (1 - \exp(-\Delta))^n \cdot \Theta_{\alpha,\delta,k}(h_{i-n}, h_{l-n}) \leqslant (1 - \exp(-\Delta))^n \cdot \Delta \tag{4.15}$$

for each $n \in \mathbb{N}$ and each pair of $i,\, l \in \mathbb{N}$ with $i,\, l \geqslant n+1$.

By Theorem 4.2, there exists $\nu \in \mathcal{P}(X)$ with $\mathcal{L}_0^*(\nu) = \nu$. Hence, we have $\langle \nu, h_n \rangle = \langle \nu, \mathbb{1} \rangle = 1$ for each $n \in \mathbb{N}_0$. Thus, by Lemma 4.5, we obtain that

$$A_{\alpha,\delta,k}(h_i, h_l) \leqslant \inf_{z \in M} \left\{ \frac{h_l(z)}{h_i(z)} \right\} \leqslant 1 \leqslant \sup_{z \in M} \left\{ \frac{h_l(z)}{h_i(z)} \right\} \leqslant B_{\alpha,\delta,k}(h_i, h_l). \tag{4.16}$$

for each pair of $i,\, l \in \mathbb{N}_0$. moreover, for each $n \in \mathbb{N}_0$, by Proposition 4.6, it follows from $h_n \in \mathcal{C}_{\alpha,\delta,\lambda k}$ that

$$\sup_{z \in M} h_n(z) \leqslant (1 + m\lambda k \cdot \operatorname{diam}(M)^\alpha) \cdot \inf_{z \in M} h_n(z) \leqslant 1 + m\lambda k \cdot \operatorname{diam}(M)^\alpha. \tag{4.17}$$

Thus, by (4.1), the operator $\mathcal{L}_0 \colon C(M) \to C(M)$ is continuous under the norm $\|\cdot\|_\infty$.

By (4.17) and (4.15), we obtain that

$$\begin{aligned} h_i(z) - h_l(z) &\leqslant h_i(z) \cdot \left(\frac{h_l(z)}{h_i(z)} - 1 \right) \leqslant (1 + m\lambda k \cdot \operatorname{diam}(M)^\alpha) \cdot (B_{\alpha,\delta,k}(h_i, h_l) - 1) \\ &\leqslant (1 + m\lambda k \cdot \operatorname{diam}(M)^\alpha) \cdot (\exp(\Theta_{\alpha,\delta,k}(h_i, h_l)) - 1) \\ &\leqslant (1 + m\lambda k \cdot \operatorname{diam}(M)^\alpha) \cdot (\exp((1 - \exp(-\Delta))^n \cdot \Delta) - 1) \end{aligned} \tag{4.18}$$

for each $z \in M$, each $n \in \mathbb{N}$, and each pair of $i,\, l \in \mathbb{N}$ with $i,\, l \geqslant n+1$. Then $\{h_n\}_{n \in \mathbb{N}_0}$ is a Cauchy sequence in the norm $\|\cdot\|_\infty$. Hence, $\{h_n\}_{n \in \mathbb{N}_0}$ converges to a function $h \in C(M)$. Note that $e^x - 1 \leqslant 3x$ for each $x \in [0,1]$. Then by (4.18), it follows from the continuity of the operator $\mathcal{L}_0$ that $\mathcal{L}_0(h) = h$ and

$$\begin{aligned} \|h_{n+1} - h\|_\infty &\leqslant (1 + m\lambda k \cdot \operatorname{diam}(M)^\alpha) \cdot (\exp((1 - \exp(-\Delta))^n \cdot \Delta) - 1) \\ &\leqslant 3(1 + m\lambda k \cdot \operatorname{diam}(M)^\alpha) \cdot (1 - \exp(-\Delta))^n \cdot \Delta \end{aligned}$$

for each integer $n \geqslant \frac{-\log(\Delta)}{\log(1 - \exp(-\Delta))}$. □

4.2. **Computability of the equilibrium state.** In this section we prove the computability of the equilibrium state $\mu_{f,\phi}$ associated to the map f and the potential ϕ satisfying the Assumptions and the Additional Assumptions.

Before the proof of this theorem, we shall do some preparations. We begin with designing an algorithm that computes the preimages of f.

Proposition 4.10. *Let $(M,\, d,\, \mathcal{S})$ be a computable metric space in which M is recursively compact. Then there exists an algorithm that satisfies the following property:*

For each M, $f(x)$, $r(x)$ satisfying items (i) to (iii) in the Assumptions, each $x_0 \in M$, each oracle $\tau \colon \mathbb{N} \to \mathbb{N}$ for x_0, and each $t \in \mathbb{N}$, this algorithm outputs $\{y_i\}_{i=1}^{\deg(f)}$ satisfying that there exists a corresponding enumeration $\{x_i : 1 \leqslant i \leqslant \deg(f)\}$ of $f^{-1}(\{x_0\})$ such that $d(x_i, y_i) < 2^{-t}$ for each integer $1 \leqslant i \leqslant \deg(f)$, after inputting the following data in this algorithm:

(i) *an algorithm computing the map $f \colon M \to M$,*
(ii) *an algorithm computing the function $r \colon M \to \mathbb{R}^+$,*
(iii) *the integers t and $\deg(f)$,*
(iv) *the oracle τ.*

As an immediate consequence of Proposition 4.10 and the computability of the exponential function, one gets the computability of the Ruelle–Perron–Frobenius operator in the following sense.

Corollary 4.11. *Let $(M,\, d,\, \mathcal{S})$ be a computable metric space in which M is recursively compact. Then there exists an algorithm that satisfies the following property:*

For each M, $f(x)$, $r(x)$, $\phi(x)$ satisfying items (i) to (iii) in the Assumptions and item (x) in the Additional Assumptions, each $x_0 \in M$, each oracle $\tau\colon \mathbb{N} \to \mathbb{N}$ for x_0, each $s \in \mathbb{N}$, and each $t \in \mathbb{N}$, this algorithm outputs a rational 2^{-t}-approximation for the value of $\mathcal{L}^s(\mathbb{1})(x_0)$, after inputting the following data in this algorithm:

(i) *an algorithm computing the map $f\colon M \to M$,*
(ii) *an algorithm computing the function $\phi\colon M \to \mathbb{R}$,*
(iii) *an algorithm computing the function $r\colon M \to \mathbb{R}^+$,*
(iv) *the integers s, t, and $\deg(f)$,*
(v) *the oracle τ.*

Now we establish the computability of the topological pressure.

Lemma 4.12. *Let $(M,\, d,\, \mathcal{S})$ be a computable metric space in which M is recursively compact, where $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$. Then there exists an algorithm that satisfies the following property:*

For each M, $f(x)$, $L(x)$, $r(x)$, σ, L, $\{U_i\}_{i=1}^n$, q, ε_ϕ, $\phi(x)$ satisfying the Assumptions and the Additional Assumptions and each $t \in \mathbb{N}$, this algorithm outputs a rational 2^{-t}-approximation for the topological pressure $P(f,\phi)$, after inputting the following data in this algorithm:

(i) *an algorithm computing the map $f\colon M \to M$,*
(ii) *an algorithm computing the function $\phi\colon M \to \mathbb{R}$,*
(iii) *an algorithm computing the function $r\colon M \to \mathbb{R}^+$,*
(iv) *the integers t and $\deg(f)$.*

Proof. We can design the algorithm following the steps below:

(1) Compute $N \in \mathbb{N}$ with $N > 2^{t+1}\log 2$.
(2) Apply Corollary 4.11 to compute and output the value of
$$v_t \approx w_t \coloneqq N^{-1}\log\bigl(\mathcal{L}^N(\mathbb{1})(s_1)\bigr)$$
with precision 2^{-t-1}.

Let us verify that v_t satisfies $|v_t - P(f,\phi)| < 2^{-t}$ for each $t \in \mathbb{N}$. To see this, it suffices to check that $|w_t - P(f,\phi)| < 2^{-t-1}$ for each $t \in \mathbb{N}$.

We set the corresponding integer computed by step (1) to be N and $k \coloneqq (m \cdot \operatorname{diam}(M)^\alpha)^{-1}$. By Theorem 4.8 and item (ix) in the Additional Assumptions, it follows from $\mathbb{1} \in \mathcal{C}_{\alpha,\delta,k}$ that $\mathcal{L}_0^N(\mathbb{1}) \in \mathcal{C}_{\alpha,\delta,\lambda k} \subseteq \mathcal{C}_{\alpha,\delta,k}$. By Theorem 4.2, we have $\bigl\langle \nu, \mathcal{L}_0^N(\mathbb{1})\bigr\rangle = 1$. Hence, by Proposition 4.6 and $k = (m \cdot \operatorname{diam}(M)^\alpha)^{-1}$, we obtain that
$$1/2 = (1 + mk\cdot \operatorname{diam}(M)^\alpha)^{-1} \leqslant \inf_{z\in M}\bigl(\mathcal{L}_0^N(\mathbb{1})(z)\bigr) \leqslant 1 \leqslant \sup_{z\in M}\bigl(\mathcal{L}_0^N(\mathbb{1})(z)\bigr) \leqslant 1 + mk\cdot\operatorname{diam}(M)^\alpha = 2.$$
Hence, we can conclude that
$$|w_t - P(f,\phi)| = \bigl|N^{-1}\log\bigl(e^{-NP(T,\varphi)}\mathcal{L}^N(\mathbb{1})(s_1)\bigr)\bigr| = \bigl|N^{-1}\log\bigl(\mathcal{L}_0^N(\mathbb{1})\bigr)\bigr| < (\log 2)/N < 2^{-t-1}. \qquad \square$$

Recall that in Theorem 4.9, we can demonstrate that the existence of eigenfunction of the normalized Ruelle–Perron–Frobenius operator.

Lemma 4.13. *Let $(M,\, d,\, \mathcal{S})$ be a computable metric space in which M is recursively compact, where $\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$. Then there exists an algorithm that satisfies the following property:*

For each M, $f(x)$, $L(x)$, $r(x)$, σ, L, $\{U_i\}_{i=1}^n$, q, ε_ϕ, $\phi(x)$ satisfying the Assumptions and the Additional Assumptions, each $x_0 \in M$ and each oracle $\tau\colon \mathbb{N} \to \mathbb{N}$ for x_0, and each $t \in \mathbb{N}$, this algorithm *outputs a rational 2^{-t}-approximation for the value of $h(x_0)^3$, after inputting the following data in this algorithm:*

(i) *an algorithm computing the map $f\colon M \to M$,*
(ii) *an algorithm computing the function $\phi\colon M \to \mathbb{R}$,*
(iii) *an algorithm computing the function $r\colon M \to \mathbb{R}^+$,*
(iv) *an algorithm computing σ, L, and ε_ϕ,*
(v) *the integers t, $\deg(f)$, and q.*
(vi) *the oracle τ*

Finally, we review [BHLS25, Theorem 5.10] in our context here.

Theorem 4.14. *Let $(X, \rho, \mathcal{S}, \{X_n\}_{n\in\mathbb{N}}, \{T_n\}_{n\in\mathbb{N}})$ be a uniformly computable system with $X_n \coloneqq X$ for each $n \in \mathbb{N}$. Assume that there exist two recursively enumerable sets K, L with $L \subseteq \mathbb{N} \times K$ and a sequence $\{Y_{n,k}\}_{(n,k)\in L}$ of uniformly lower semi-computable open sets in $(X, \rho, \mathcal{S})$ such that $Y_{n,k}$ is admissible for T_n, and $X = \bigcup_{(n,k)\in L_n} Y_{n,k}$, where $L_n \coloneqq \{(n, k) \in L : k \in K\}$ for each $n \in \mathbb{N}$. Suppose $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly computable functions satisfying that $\mathcal{E}_0(T_n, \phi_n) = \{\mu_n\}$. Moreover, assume that $\{J_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly lower semi-computable functions $J_n\colon X \to [0, +\infty)$ satisfying the following properties:*

(i) *There exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ of continuous functions $u_n\colon X \to \mathbb{R}$ such that for each $x \in X$,*
$$J_n(x) = \exp(P(T_n, \phi_n) - \phi_n(x) + u_n(T_n(x)) - u_n(x)).$$
(ii) $\sum\limits_{y\in(T_n)^{-1}(x)} \frac{1}{J_n(y)} = 1$ *for each $x \in X$.*

Then $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly computable measures.

Proof. By Lemma 4.13 and Theorem 4.14, we obtain Theorem 1.2. □

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[3] Here $h \in C(M)$ satisfies that $\mathcal{L}(h) = \exp(P(f, \phi)) \cdot h$. The existence of such function is given by Theorem 4.9.

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